Exhibit No. ___ (JHV-3)
Docket No. UT-023003

A Note on the Evaluation of Cancellable Operating Leases

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■ Many central theoretical issues on long-term leasing were settled by Miller and Upton [8], Lewellen, Long and McConnell [6], and Myers, Dill and Bautista [9].[1] Issues of clarification and implementation can be found in Levy and Sarnat [5]. The following paper extends the analysis of lease contracts to include cancellable operating leases.

For expositional purposes lease contracts can be divided into two broad categories: 1) pure financial leases and 2) operating leases. Pure financial leases are assumed to be perfect substitutes for debt capital because they are not cancellable without bankruptcy and they are fully amortized. On the other hand, operating leases are riskier from the lessor's point of view because they may be cancelled at the option of the lessee and cannot (by law) be fully amortized.

The first part of the paper provides a brief review of the analysis of pure financial leases. The second part solves the problem of evaluating cancellable operating leases by using the Cox, Ross and Rubinstein [2] binomial option pricing method. From the lessor's point of view a cancellable operating lease is equivalent to a pure financial lease minus an American put option with a (non-stochastic) declining exercise price. The expected rate of return on a cancellable lease is shown to be higher than the rate on a pure financial lease.

The Analysis of Pure Financial Leases

Pure financial leases are assumed to be perfect substitutes for debt. The lessee takes the before-tax rental rate, L_t, as an input in making a comparison between leasing and borrowing. The analysis involves the following differential cash flows:

a. A cash saving amounting to the dollar amount of the investment outlay, I, which the firm does not have to incur if it leases.

b. A cash outflow amounting to the present value of the after-tax lease dollars which must be paid out, $PV[(1-\tau_c)L_t]$.

We wish to thank Dan Galai, Robert Geske and Kuldeep Shastri for their helpful comments.

[1]The distinction between long-term and short-term leases is not trivial. Short-term leases such as hotel room rentals are probably more efficient than buying for a day simply because of transaction cost differences. However, the effect of such frictions is minimized for long-lived contracts.

c. The present value of the opportunity cost of the lost depreciation tax shield, $PV(\tau_c dep_t)$.

d. The present value of the *change* in the interest tax shield on debt which is displaced by lease financing, $PV[\tau_c \Delta(rD_t)]$, where D_t is the remaining principal of displaced debt in period t, and r is the coupon rate.

These four terms, when discounted at the proper rate, give the net present value (NPV) of the lease contract to the lessee. If the NPV (to lessee) > 0 the lease will be accepted.

$$NPV\ (lessee) = I - PV[(1-\tau_c)L_t] - PV\ [\tau_c dep_t] - PV[\tau_c \Delta(rD_t)] \quad (1)$$

Because this definition of cash flows explicitly includes the tax shield of displaced debt in the numerator of the present value equation, the cash flows should be discounted at the before-tax cost of capital. The before-tax cost of *debt* capital, k_d, is relevant because the lease contract is a perfect substitute for debt. It has the same risk. Therefore, we have

$$NPV\ (lessee) = I - \sum_{t=1}^{N} \frac{(1-\tau_c)L_t + \tau_c dep_t + \tau_c \Delta(rD_t)}{(1+k_d)^t} \quad (2)$$

If correct, this approach should show the lessee to be indifferent to the contract (*i.e.*, NPV (lessee) = 0) when the lessor's minimum lease fee is substituted into the equation. The computation is fairly cumbersome because the displaced tax shield, $\tau_c \Delta(rD_t)$, changes each period.

Myers, Dill and Bautista [9] and Levy and Sarnat [5] have shown that an equivalent approach is to account for the interest tax shield by discounting at the after-tax cost of debt and eliminating the third term from the numerator of the righthand side of Equation (2). For constant lease payments, Equations (2) and (3) are equivalent.

$$NPV\ (lessee) = I - \sum_{t=1}^{N} \frac{(1-\tau_c)L_t + \tau_c dep_t}{[1+(1-\tau_c)k_d]^t} \quad (3)$$

Note that from the lessor's point of view k_d is the lending rate on debt capital. It is the lessor's weighted average cost of capital, WACC (lessor), grossed up by the lessor's effective marginal tax rate.[2]

[2]For reasons why the marginal effective tax rate may be different from the corporation's marginal nominal tax rate see Miller [7] and DeAngelo and Masulis [3].

$$k_d = \frac{WACC(lessor)}{(1-\tau_c)} \quad (4)$$

Therefore, when discounting the cash flows of Equation (3) from the lessor's point of view, we have

$$NPV(to\ lessor) = -I + \sum_{t=1}^{N} \frac{L_t(1-\tau_c) + \tau_c dep_t}{(1+WACC)^t} \quad (5)$$

where $WACC(lessor) = (1-\tau_c)k_d$. The equivalence of Equations (3) and (5) demonstrates that the financing decision is the same from either the lessee's or lessor's point of view. Also, it is worth mentioning that the lessee's indifference to the contract will result only when all terms in Equations (3) and (5) are symmetrical. Especially important are the effective tax rates of the lessor and lessee. Lewellen, Long and McConnell [6] have shown that with different effective tax rates for the lessor and lessee the lease may have positive net present values for both parties.

The Evaluation of Operating Lease Contracts

Operating leases are different from pure financial leases in two important ways. First, and most important, they may be cancelled at the option of the lessee. From the point of view of the lessee, capital employed under operating lease contracts becomes a variable cost (rather than a fixed cost) because the lease may be terminated (sometimes requiring a penalty to be paid) and the leased asset may be returned whenever economic conditions worsen. This is like having equipment that can be laid off. From the lessor's point of view, operating leases are obviously riskier than financial leases. A financial lease, like a loan, is secured by all of the firm's assets. An operating lease is not. The second difference between operating and financial leases is that operating leases enable the lessor to capture the salvage value of the asset.

The duration of an operating lease is usually several years on business office equipment, computers, buildings, and trucks. The contracts are not renegotiated during their term. However, they can usually be cancelled at the option of the customer (sometimes with and sometimes without penalty). For example, the wording in an IBM contract is: ". . . the customer may, at any time after installation, discontinue a processor complex unit upon three months prior written notice, or discontinue any other machine or any field removable feature or request a field removable downgrade upon one month's written notice" [subject to the payment of termination charges].

What are the sources of risk to a lessor who contemplates extending an operating lease? We shall discuss two categories of risk. The first category of risk reflects fluctuations in the economic value of the asset over time. These changes in value result from the uncertain economic rate of depreciation of the asset and from general price level and interest rate uncertainty. The economic rate of depreciation is determined by the value of the asset in alternative uses and from the competition of substitutes. Changes in value will reflect obsolescence as well as physical deterioration. This may be termed *replacement cost risk*. The uncertainty of the salvage value of the asset is a special case of this first category of risks related to the economic value of the asset. Our intent is to define replacement cost risk as the generic term for fluctuations in the economic value of the asset resulting from uncertainties such as obsolescence costs and unanticipated changes in the general price level and interest rates.

A second category of risk relates to the characteristics of the lessee and we shall argue that they are of no special concern to the lessor. (The reason is discussed below.) Related to the performance of the lessee is a *revenue risk*. This is the risk that the lease will be cancelled because the lessee's revenues from the asset fall enough so that the present value of the lease payments exceeds the present value of continued use of the asset.

Another source of risk related to the behavior of the lessee is the *risk of default*. Default is an involuntary breach of the lease contract. It is common to both financial leases and operating leases. Therefore, we shall assume that the lessor's lending rate, k_d, is already adjusted to compensate for default risk.

The usual approach to the operating lease problem is to separate each of the different components of risky cash flow and discount them at the "appropriate" risk-adjusted discount rate.[3] The type of formula often used is:

NPV(to lessee) =

$$I - \sum_{t=1}^{N} \frac{L_t(1-\tau_c)}{(1+k_d^*)^t} - \sum_{t=1}^{N} \frac{\tau_c dep_t}{(1+k_d^*)^t} - \tau_i I - \frac{MV}{(1+k_1)^N} + \frac{t_g(MV-BV)}{(1+k_1)^N} - \sum_{t=1}^{N} \frac{O_t}{(1+k_2)^t} \quad (6)$$

where

$k_d^* = (1-\tau_c)k_d$ = the after-tax cost of debt capital;

$\tau_i I$ = the investment tax credit forgone by the lessee;

MV = the salvage value (market value) of the asset when the lease contract expires in year N;

k_1 = the risk-adjusted after-tax discount rate "appropriate" to salvage risk;

$t_g(MV-BV)$ = the capital gains tax on the difference between the salvage value and the book value;

O_t = the value of operating maintenance in period t;

k_2 = the risk-adjusted after-tax discount rate "appropriate" to the maintenance costs.

While this approach is useful in pointing out the different risks that exist, the practitioner is forced to use ad hoc rules of thumb when attempting to estimate the various risk-adjusted discount rates needed to solve Equation (6).[4] Another approach is suggested below.

Of the types of risk mentioned above, only replacement cost risk (including salvage value risk) and default risk are borne by the lessor. Default risk is compensated in the lending rate, k_d, and shall not be discussed. Revenue risk is irrelevant to the lessor because it is borne by the lessee when he makes his investment decision. To show why this is so, assume for the moment that the replacement cost and salvage value of the asset are known with certainty. Still, the lessee may cancel an operating lease if the present value of the after-tax operating cash flow from his use of the leased asset falls below the present value of the future lease obligations. Even so, the lessor will be indifferent to the cancellation because, given no uncertainty about the replacement or salvage value of the asset, a lease contract can always be constructed so that the replacement value of the asset is equal to the value of the remaining lease payments. The payoffs to the lessor are:

Payoff to lessor (given no replacement cost risk)

$$= \begin{cases} \text{PV (lease payments)} & \text{if NPV (project)} \geq 0 \\ \text{PV (asset)} & \text{if NPV (project)} > 0 \end{cases}$$

Given no replacement cost uncertainty a contract can be written so that

$$\text{PV (lease payments)} = \text{PV (asset)}$$

[3]For example, see [10].

[4]Maintenance contracts for leased assets are separable from the lease contract itself and can be priced separately. Therefore, we ignore maintenance cost cash flows when we discuss the operating lease contract.

for any point in time. Thus, the lessor is indifferent to revenue uncertainty.[5]

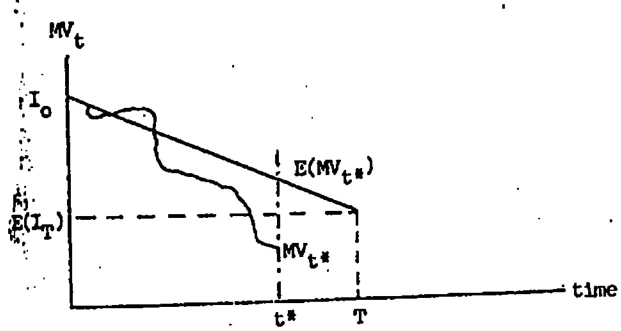
Given the irrelevance of revenue uncertainty, we can proceed to discuss the effect of uncertain replacement costs (including uncertain salvage value). Exhibit 1 shows how the market value of the leased asset may change over time. The downward-sloping solid line is the expected decline in the asset's value due to anticipated inflation, wear and tear, and obsolescence. Note that the value of the asset is expected to decline from $\$I_0$ to $E(\$I_T)$ over the life of the contract, T years. The expected salvage value is $E(I_T)$. It is reasonable to assume that the value of the asset never falls below zero. Given replacement cost uncertainty, the actual value of the asset at any time $t^* \leq T$ may be greater or less than expected. The particular situation illustrated at t^* in Exhibit 1 shows that if the value of the asset, MV_{t^*}, falls far enough below its expected value, $E(MV_{t^*})$, then the lessee can improve his position by cancelling the lease, returning the leased asset, and leasing a more efficient replacement to do the same job at lower cost. The option to terminate the lease is an American put held by the lessee. The value of the put will be implicit in the lease fees.[6]

The present value of the relevant American put, P_A, is derived in Appendix A following the assumption of a binomial stochastic process. (Cf. Cox, Ross and Rubinstein [2]). The expected replacement cost of the asset is assumed to decline in a straight line at the rate

[5] This point is also made in Miller and Upton [8]. Implicitly, it is understood that if the original lease is cancelled the lessor immediately places the equipment on lease again.

[6] *Ex ante*, the lessor will be seen to charge for possible actions by the lessee under alternate states of nature (captured in the *ex ante* probability distribution). *Ex post*, of course, the asset may decline in value so that the lessee will return the asset. The lessor then must either a) sell the asset at market value or b) lease it again at a lower rate. Both possibilities are reflected in the price of the American put in the *ex ante* analysis (see Equation 7).

Exhibit 1. Replacement Cost Uncertainty



$(1-\theta)$ in each period. For convenience, we assume that the lease contract is written so that the present value of the remaining lease fees is equal to the expected replacement value of the asset in each time period. Hence the option is written at-the-money.

If the lease contract is written so that the exercise price of the implied put declines at a rate slower than the expected economic depreciation, then the probability of cancellation increases. If there are any significant transactions costs such as installation and removal and resale expenses, then frequent cancellation is undesirable. The opposite situation occurs when the exercise price declines faster than expected economic depreciation. The likelihood of early exercise decreases and so does the implied value of the cancellation feature. If there are costs to negotiating the terms of the cancellation feature, then the value of the cancellation option must exceed negotiation costs. There may well be an optimal relationship between the rate of decline in the exercise price and the expected economic depreciation of the asset. No matter what it is, Equation (7) will provide a numerical solution for the value of the American put implied in the cancellation clause. Modifications in this assumption do not materially alter the form of the option pricing equation. The exercise price, X, for the American put written on the replacement cost of the asset is the present value of the lease payments represented by the solid line in Exhibit 1. Since the lease payments include repayment of the expected economic depreciation of the asset, $(1-\theta)E(MV_t)$, we have to price the value of an American put for a case in which the exercise price declines at a non-stochastic rate equal to the expected decline in the value of the asset (analogous to a non-stochastic dividend payment). The present value of the American put is:[7]

$$P_A = \text{MAX}\{X-V,\ [pP_d + (1-p)P_u] \div r_f\} \qquad (7)$$

where

$$P_d = \text{MAX}\{\theta X - d\theta V,\ [pP_{dd} + (1-p)P_{ud}] \div r_f\};$$
$$P_u = \text{MAX}\{\theta X - u\theta V,\ [pP_{ud} + (1-p)P_{uu}] \div r_f\};$$
$$p = \frac{(u-1) - (r_f-1)/\theta}{u-d},\quad (1-p) = \frac{(r_f-1)/\theta + (1-d)}{u-d}.$$

Equation (7) may be solved iteratively in order to provide a numerical solution for any American put option where the exercise price on the option declines at a non-stochastic rate equal to the *ex ante* expected decline in the value of the asset. If the depreciation rate $(1-\theta)$ is zero, then Equation (7) reduces exactly to the

[7] The notation used in Equation (7) is detailed in the appendix.

numerical solution of an American put with constant exercise price, derived by Cox, Ross and Rubinstein [2]. As the anticipated economic life of the asset becomes shorter (*i.e.*, as it depreciates faster), the value of the put decreases relative to its counterpart — an American put with fixed exercise price. The put implied by the lease's cancellation clause differs from a regular American put because its exercise price decreases at a predetermined rate. Because the decreasing exercise price is linked to the anticipated rate of economic depreciation, it follows that the put is worth less as the expected life of the underlying asset is shorter.

The effect of the put on the lease fees will be to increase them with 1) greater uncertainty in the replacement cost of the leased asset, 2) decreases in the risk-free discount rate, and 3) a lower expected rate of depreciation over the life of the lease contract. The first two effects are obvious and the third effect makes sense when one realizes that we are talking about the marginal change in lease fees caused by the cancellation option. The level of lease fees will decrease as the expected rate of economic depreciation decreases, but the cancellation option has greater cost to the lessor as the life of the asset increases.

An American put written on a lease contract and modeled as in Equation (7) will capture the value of the cancellation clause in an operating lease. The value of the put will depend on the following variables:

$$P_A = f(\overset{+}{I}, \overset{+}{\sigma^2_{MV}}, \overset{-}{r_f}, \overset{\pm}{T}, \overset{+}{X}, \overset{+}{\theta}) \tag{8}$$

where I = the initial cost of the leased asset;
σ^2_{MV} = the instantaneous variance of the market value of the asset (for annual binomial outcomes $u = e^{\sigma}$, where σ is the annual standard deviation of asset returns);
r_f = one plus the risk-free rate for assets of maturity T;
T = the number of time periods before the option expires;
X = the initial exercise price of the option ($X = I$);
$1\text{-}\theta$ = the annual rate of anticipated straight-line depreciation in the value of the asset.

The sign of the partial derivative of the value of the put with respect to each of the variables is given above Equation (8).

The following numerical example shows how the lessor will increase his required lease payments if a lease contract is cancellable. Assume that a $10,000 asset is expected to have a three-year economic life and depreciate an equal amount each year (*i.e.*, $\theta = .667$). However, its value may be 50 percent higher or lower than expected at the end of a given year (*i.e.*, $u = 1.50$, $d = .667$, $\sigma = .405$). The lessor has a tax rate of 40 percent and will write a two year lease.[8] If the lease contract were a strict financial lease, it would require a 10 percent before-tax rate of return (*i.e.*, $k_d = 10\%$). The salvage value is uncertain and requires a 16% risk-adjusted rate of return. For simplicity we ignore capital gains taxation on the salvage value and investment tax credits. Using our prior definitions of the variables we can write the competitive present value of a non-cancellable lease to the lessor as follows:

$$0 = -I + \sum_{t=1}^{2} \frac{(1-\tau_c)L_t + \tau_c dep_t}{[1+(1-\tau_c)k_d]^t} + \frac{E(MV)_t}{(1+k_s)^2}. \tag{9}$$

Substituting in the appropriate values, and solving for the competitive lease fee we have

$$0 = -10{,}000 + \sum_{t=1}^{2} \frac{(1-.4)L_t + .4(3333)}{[1-(1-.4).10]^t} + \frac{3333}{(1.16)^2}$$

$$0 = -10{,}000 + .6L_t\text{PVIF}_a\,(6\%,\ 2\text{ yrs.}) + .4(3333)\text{PVIF}_a\,(6\%,\ 2\text{ yrs.}) + 3333(.743)$$
$$0 = -10{,}000 + .6L_t(1.833) + .4(3333)(1.833) + 3333(.743)$$
$$L_t = \$4{,}619$$

Next, we want to determine the competitive lease payments assuming that the above contract is a cancellable operating lease. Equation (9) must be modified by subtracting the present value of the American put option. The new valuation equation is

$$0 = -I + \sum_{t=1}^{2} \frac{(1-\tau_c)L'_t + \tau_c dep_t}{(1+(1-\tau_c)k_d)^t} + \frac{E(MV)_t}{(1+k_s)^2} - P_A. \tag{10}$$

[8]For simplicity we will assume that the lessor and the lessee have the same effective tax rate. Differential tax rates do not affect the value of the cancellation clause.

The value of the put (per dollar value of the asset) is given in Exhibit A-4 as .085. Solving for the operating lease fee we have

$$0 = -10{,}000 + \sum_{t=1}^{2} \frac{(1-.4)L'_t + .4(3333)}{(1+(1-.4).10)^t} + \frac{3333}{(1.16)^2} - .085(10{,}000)$$

$$0 = -10{,}000 + .6L'_t\,(1.833) + .4(3333)\,(1.833) + 3333(.743) - 850$$

$$L'_t = \$5{,}392$$

The lease fee has increased considerably to reflect the extra risk of possible early cancellation of the operating lease.

If a lessee takes the lease fee as an input and tries to compute an internal rate of return (IRR) on the contract without considering the American put, then there will be a considerable upward bias in the IRR. Using the above lease fee the computation would be

$$0 = I - \sum_{t=1}^{2} \frac{(1-\tau_c)L'_t + \tau_c dep_t}{(1+IRR)^t} - \frac{E(MV)}{(1+k_1)^2}$$

$$0 = 10{,}000 + \sum_{t=1}^{2} \frac{(1-.4)\,(5392) + .4(3333)}{(1+IRR)^t} - \frac{3333}{(1.16)^2}$$

$$0 = 10{,}000 - 4568.4\,PVIF_a\,(IRR\%,\ 2\text{ yrs.}) - 2476$$

$$PVIF_a\,(IRR\%,\ 2\text{ yrs.}) = \frac{-7524}{-4568.4} = 1.647$$

$$IRR \cong 14\%$$

The management of the lessee firm would be mistaken to compare the 14 percent before-tax rate of return with the 10 percent before-tax cost of debt capital. The two rates are not comparable because the cancellable operating lease is riskier than its non-cancellable financial lease counterpart.

Frequently the lease may be cancelled only if a lump-sum penalty, F, is paid to the lessee. The penalty reduces the value of the cancellation clause for the lessee. Numerically, the effect of the penalty can be estimated by subtracting the fee from the exercise price in Equation (7). This is shown below where P^*_A is the present value of the cancellation clause given a cancellation fee, F:

$$P^*_A = MAX\{(X-F) - V_t[pP_d + (1-p)P_u] \div r_f\}$$

where

$$P_d = MAX\{(\theta X-F) - d\theta V_t[pP_{dd} + (1-p)P_{ud}] \div r_f\};$$

$$P_u = MAX\{(\theta X-F) - u\theta V_t[pP_{ud} + (1-p)P_{uu}] \div r_f\};$$

$$p = \frac{(u-1) - (r_f-1)/\theta}{u - d},$$

$$(1-p) = \frac{(r_f-1)/\theta + (1-d)}{u - d}.$$

Summary

If the lease is a pure financial lease, it is a perfect substitute for debt and we show that the appropriate discount rate for the leasing cash flows (before interest charges) is the after-tax cost of debt capital. On the other hand, if the lease contract is a cancellable operating lease, it is not a perfect substitute for debt capital and some higher discount rate is appropriate. This rate may be obtained by first computing the present value of an American put with an exercise price that declines at the same rate as the expected decline in the market value of the leased asset. The declining exercise price is necessary so that at any time the expected value of the future lease payments is equal to the expected market value of the depreciating asset. An example shows that the internal rate of return on an operating lease will be greater than on the comparable pure financial lease. However, the apparent higher internal rate reflects the value of the put included in the cancellation clause of an operating lease.

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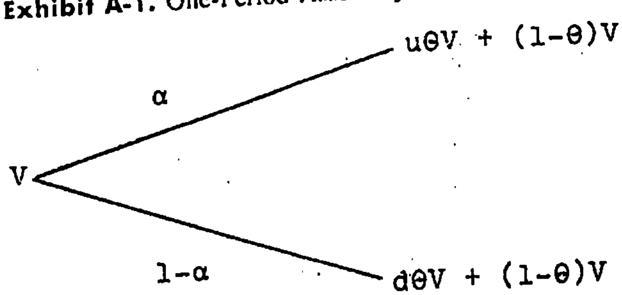
Appendix A. Derivation of the Price of an American Put Option Where the Exercise Price Declines at a Non-stochastic Rate Equal to the Expected Decline in the Asset's Value

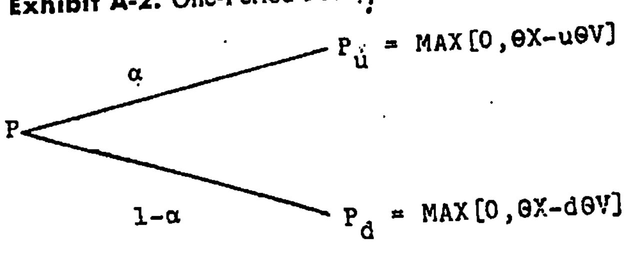
Let V be the current value of an asset that is expected to decline in value in a straight-line fashion at the rate of $(1-\theta)$ percent per time period. The value of the asset at the end of one period will be $u\theta V$ (where $u > 1$) with probability α and $d\theta V$ (where $d = 1/u$) with probability $1-\alpha$. Thus, changes in the value of the asset are described by a binomial process. Furthermore, the asset pays a "dividend" of $(1-\theta)V$ with certainty. Exhibit A-1 shows the one-period payoffs from holding the asset.

Exhibit A-1. One-Period Asset Payoffs



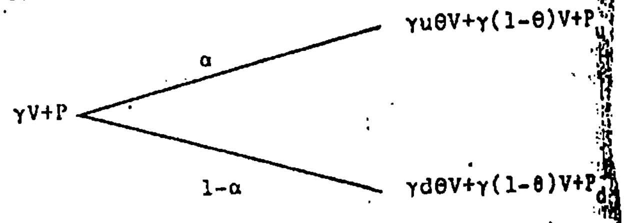
A put option written on the asset has the payoffs shown in Exhibit A-2.

Exhibit A-2. One-Period Put Option Payoffs



Note that the exercise price, X, has declined by an amount exactly equal to the certain dividend, $(1-\theta)V$, assuming that the option is written at the money, *i.e.*, if $V = X$. A riskless hedge can be created by purchasing a fraction, γ, of the risky asset and buying one put written on the asset. The one-period payoffs of the hedge portfolio are given in Exhibit A-3.

Exhibit A-3. One-Period Payoffs on the Hedge Portfolio



In order to prevent riskless arbitrage we require that one plus the one-period risk-free rate, r_f, lie between the up and down movements in the binomial process, *i.e.*, $d<r_f<u$. In order to find the ratio, γ, which creates a riskless hedge, equate the end-of-period payoffs from the hedge portfolio

$$\gamma u\theta V+\gamma(1-\theta)V+P_u = \gamma d\theta V+\gamma(1-\theta)V+P_d \quad \text{(A-1)}$$

where
$$\gamma = \frac{P_d - P_u}{\theta V(u-d)}.$$

Note that since $P_u<P_d$, we are long in the risky asset, *i.e.*, $\gamma>0$. Next, use the fact that the hedge portfolio must earn the risk-free rate to write

$$r_f(\gamma V+P) = \gamma u\theta V + \gamma(1-\theta)V + P_u. \quad \text{(A-2)}$$

Substituting in the value of γ and solving for P, we have

$$P = \frac{P_d\left[\dfrac{(r_f-1)/\theta + (1-d)}{u-d}\right] + P_u\left[\dfrac{(u-1) - (r_f-1)/\theta}{u-d}\right]}{r_f} \quad \text{(A-3)}$$

Now, let

$$p = \frac{(u-1) - (r_f-1)/\theta}{u-d} \quad \text{and}$$

$$(1-p) = \frac{(r_f-1)/\theta + (1-d)}{u-d}.$$

Then formula (A-3) becomes

$$P = [pP_d + (1-p)P_u] \div r_f.$$

Note that $p + (1-p) = 1$. Furthermore, if $\theta = 1$, so that the asset does not depreciate, then our formula A-3 is identical to that of Cox, Ross and Rubinstein [2].

the economic value of the asset is expected to decline, then $\theta<1$ and we also require that $\theta>(r_f-1)/(u-1)$ in order that $0\leq p\leq 1$. In other words, the asset cannot be expected to depreciate so rapidly that riskless arbitrage becomes possible.

If the put is an American put, P_A, we must allow for the possibility that the put may be exercised early. Therefore, the pricing equation (A-3) for the one-period put must be rewritten as

$$P_A = MAX\{X\text{-}V, [pP_d + (1-p)P_u] \div r_f\}. \quad (A\text{-}4)$$

If $r_f>1$ (and it is), it is certainly possible that early exercise may be optimal.[9] Suppose that V is sufficiently low so that $X>uV>dV$. In this event, $P_d = \theta X - d\theta V$ and $P_u = \theta X\text{-}u\theta V$. Substituting these values into (A-4) we have

$$\begin{aligned} P_A &= MAX\{X\text{-}V, [p(\theta X-d\theta V) \\ &\quad + (1-p)(\theta X-u\theta V)] \div r_f\} \\ &= MAX\{X\text{-}V, \frac{\theta X}{r_f} - \theta V[p\frac{d}{r_f} + (1-p)\frac{u}{r_f}]\}. \end{aligned}$$

Early exercise is advantageous whenever

$$X\text{-}V > \frac{\theta X}{r_f} - \theta V[p\frac{d}{r_f} + (1-p)\frac{u}{r_f}].$$

Substituting in the values of p and $(1-p)$ this condition becomes

$$\theta < \frac{r_f X\text{-}V}{X\text{-}V}$$

and since we know that $X>V$ and $r_f>1$, early exercise will be optimal if $\theta<1+\frac{X(r_f-1)}{X\text{-}V}$. This shows that for $r_f>1$, $\theta<1+\frac{X(r_f-1)}{X\text{-}V}$ and V sufficiently low, it pays the put-holder to exercise his put early to receive X-V. There is always a critical value for the underlying risky asset V^* such that if $V<V^*$ the put should be exercised immediately.

From equation (A-4) we can move one period back to derive the value of a two-period American put:

$$P_A = MAX\{X\text{-}V, [pP_d + (1-p)P_u] \div r_f\} \quad (A\text{-}5)$$

where

$$P_d = MAX\{\theta X\text{-}d\theta V, [pP_{du} + (1-p)P_{dd}] \div r_f\} \quad (A\text{-}6)$$
$$P_u = MAX\{\theta X\text{-}u\theta V, [pP_{ud} + (1-p)P_{uu}] \div r_f\}$$

and at the expiration date,

$$P_{dd} = MAX[0, (2\theta-1)X-d^2(2\theta-1)V] \quad (A\text{-}7)$$
$$P_{ud} = MAX[0, (2\theta-1)X-ud(2\theta-1)V]$$
$$P_{uu} = MAX[0, (2\theta-1)X-u^2(2\theta-1)V].$$

Equations A-5 through A-7 may be solved iteratively in order to compute the exact current value of a two-period American put. For example, the value of A-7 determines the value of A-6 which in turn determines the value of A-5.

Exhibit A-4 compares the prices of a "regular" two-period American put and a two-period American put written on the value of an asset which declines at the rate of 33 percent per year. Note that the options are assumed to be written at-the-money because we assume that an operating lease can be cancelled even at the first instant by returning the equipment at its initial market value. The price of the put written on the asset with depreciating value is always less than the price of the corresponding American put written on the same asset without depreciation. Thus we see that the value of the "special" American put whose value has been derived in this appendix is a function of six parameters.

$$P_A = f(V,X,r_f,T,\sigma_v,\theta). \quad (A\text{-}8)$$

The first five parameters are the usual Black-Scholes parameters and have the usual partial derivatives. In addition, the expected depreciation of the asset is relevant and $\delta P_A/\delta\theta>0$.

Exhibit A-4. American Put Comparison

Prices of Two-Period "Regular" American Puts

r_f \ u	1.3	1.5	1.7	1.9
1.1	.079	.145	.202	.251
1.3	*	.061	.143	.212
1.5	*	*	.049	.092
1.7	*	*	*	.041

Prices of Two-Period American Puts on an Asset which Declines in Value

r_f \ u	1.3	1.5	1.7	1.9
1.1	.040	.085	.124	.157
1.3	*	.009	.048	.080
1.5	*	*	†	.023
1.7	*	*	*	†

Assumptions:
1. X = V = 1.0, i.e., the lease option is written at-the-money.
2. 0 = .667, assumes three-year straight-line depreciation
3. u = 1/d, assumes proportional up and down movements in value, V.
4. The exercise price on the option decreases at the rate (1-0) percent per period.

*When the condition $d<r_f<u$ is violated, there is no option price because of riskless arbitrage opportunities.
†The condition $0>(r_f-1)/(u-1)$ is violated.

[9] If the option is written at-the-money, exercise at the beginning of the first period will not be optimal. However, for any later time period V may be low enough to make early exercise optimal.