Note

Biases in Arithmetic and Geometric Averages as Estimates of Long-Run Expected Returns and Risk Premia

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The empirically documented presence of negative autocorrelation in long-horizon common stock returns magnifies the upward (downward) bias inherent in the use of arithmetic (geometric) averages as estimates of long-run expected returns and risk premia. Failure to account for this autocorrelation can lead to incorrect project accept/reject decisions. Through simulations, we show that a horizon-weighted average of the arithmetic and geometric averages contains a smaller bias and is a more efficient estimator of long-run expected returns.

■ Consider an investment project with an average life (duration) of N months. What rate should be used to discount this project's expected cash flows? In particular, suppose the required return on the N-month investment project is based on a *market* equity-risk premium, that is, the difference between the future expected return on the market index and the risk-free rate of interest. Since risk premia are not constant (Brigham, Shome, and Vinson, 1985; Harris, 1986; Harris and Marston, 1992; Maddox, Pippert, and Sullivan, 1995; and Brennan, 1997) and can depend on the choice of measurement period, averaging method, or portfolio weighting (Carleton and Lakonishok, 1985), how should the historical monthly market return data be used to compute the risk premium? In practice, the arithmetic and geometric average of monthly returns are used as a proxy for determining the future expected N-month market return.[1]

Brealey and Myers (1991) argue that if monthly returns are identically and independently distributed, then the arithmetic average of monthly returns should be used to estimate the long-run expected return. However, the empirical evidence from Fama and French (1988a, 1988b), Lo and MacKinlay (1988), and Poterba and Summers (1988) suggests that there is significant long-term negative autocorrelation in equity returns and that historical monthly returns are not independent draws from a stationary distribution. Based on this evidence, Copeland, Koller, and Murrin (1994) argue that the geometric average is a better estimate of the long-run expected return. Thus, as noted by Fama (1996), when expected returns are autocorrelated, compounding a sequence of one-period returns is problematic for project valuation.

In this paper, we examine the biases obtained by using the arithmetic or geometric sample averages of single-period returns to assess the long-run expected rates of return when there is both a time-varying and a stationary component in those returns. To do this, we adopt the analytical framework outlined in Blume (1974). We find that for long-run expected return and risk premium, the arithmetic average produces an

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[1]Alternatively, in deriving the cost of equity estimates, Harris (1986) and Harris and Marston (1992) employ the Discounted Cash Flow (DCF) model, which uses a consensus measure of financial analysts' forecasts of earnings growth as a proxy for investor expectations. Although this alternative is appealing, Timme and Eisemann (1989) caution that it requires a judicious choice of the weight assigned to each forecast to construct the consensus forecast. Otherwise, the DCF model can generate a risk-adjusted discount rate that contains estimation risk and requires an adjustment such as that outlined in Butler and Schachter (1989).

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estimate that is too high relative to the true mean, and that the geometric average produces an estimate that is too low. The magnitude of upward and downward bias is proportional to the total variance underlying the asset's return, and to the length of the investment horizon (N months) relative to the length of the historical sample period ($T \geq N > 1$). In addition, we confirm Blume's finding that there are significant biases associated with the use of the arithmetic and geometric averages, even when returns are independently and identically distributed each period. Finally, simulation results show that the horizon-weighted average of the arithmetic and geometric averages proposed by Blume is less biased and more efficient than alternative estimates.

I. The Bias in the Arithmetic and Geometric Averages

Here, we describe the return generating process and derive the biases in the arithmetic and geometric averages.

A. Return Generating Process

Let R_t denote a one-period total return over a time interval of length dt. Specifically,

$$R_t = 1 + r_t dt = 1 + \mu_t dt + \varepsilon_t \sqrt{dt} \tag{1}$$

where $r_t dt$ is the net return for period t = 1,2,....,T; $\mu_t dt$ is the conditional mean, and the deviations from the conditional mean, $\varepsilon_t \sqrt{dt}$ are independently and identically distributed over time with mean zero and variance $\sigma_\varepsilon^2 dt$. Further, assume that the conditional mean $\mu_t dt$ is distributed as follows. For t = 1, the conditional mean is

$$\mu_1 dt = \mu dt + \eta_1 \sqrt{dt} \tag{2}$$

where μdt is the unconditional mean. For t = 2,3,....,T, the conditional mean follows a mean-reverting process around the unconditional mean:

$$\mu_{t+1} dt = \mu dt + \rho(\mu_t dt - \mu dt) + \eta_{t+1} \sqrt{dt} = (1 - \rho)\, \mu dt + \rho \mu_t dt + \eta_{t+1} \sqrt{dt} = \mu dt + \Sigma_{j=1}^{t+1} \rho^{t-j+1}\, \eta_j \sqrt{dt} \tag{3}$$

where the single-period autocorrelation between conditional means, $\rho \leq 0$, captures the time variation in expected returns, and $\eta_t \sqrt{dt}$ are independently and identically distributed random variables with mean zero and variance $\sigma_\eta^2 dt$. From Equations (1) through (3) it follows that

$$r_t dt = \mu dt + \varepsilon_t \sqrt{dt} + \Sigma_{i=1}^{t} \rho^{t-i} \eta_i \sqrt{dt} = \mu dt + \nu_t \sqrt{dt} \tag{4}$$

for all t. The return generating process described by Equation (4) is consistent with that used by Fama and French (1988a) to document significant negative autocorrelations in long-horizon returns.[2] The unconditional mean, $E(r_t dt)$, is μdt. The unconditional variance, $Var(r_t dt)$, is $[(1-\rho^{2T})/(1-\rho^2)]\sigma_\eta^2 dt + \sigma_\varepsilon^2 dt$ for a finite T, and $[1/(1-\rho^2)]\sigma_\eta^2 dt + \sigma_\varepsilon^2 dt$ as $T \rightarrow \infty$.

B. The Bias in the Arithmetic Average

From a sample of T observations, we compute the arithmetic average, R_A, as:

$$R_A = 1 + r_A dt = 1 + \mu dt + T^{-1} \Sigma_{t=1}^{T} \nu_t \sqrt{dt} \tag{5}$$

and the estimated N-period return, $R_A^N = (1 + r_A dt)^N$,

$$R_A^N = (1 + \mu dt + T^{-1} \Sigma_{t=1}^{T} \nu_t \sqrt{dt})^N \tag{6}$$

In addition, applying the expected value operators to Equation (6) yields:

$$E(R_A^N) = E(1 + \mu dt + T^{-1} \Sigma_{t=1}^{T} \nu_t \sqrt{dt})^N \tag{7}$$

Since $(1 + \mu dt + T^{-1} \Sigma_{t=1}^{T} \nu_t \sqrt{dt})^N$ is a convex function of $T^{-1} \Sigma_{t=1}^{T} \nu_t \sqrt{dt}$, it follows by Jensen's inequality that for N > 1, the arithmetic average is biased upward:

$$E(R_A^N) > (1 + \mu dt + E(T^{-1} \Sigma_{t=1}^{T} \nu_t \sqrt{dt}))^N > (1 + \mu dt)^N \tag{8}$$

Further, by taking a Taylor series expansion of $E(R_A^N)$ around $(1 + \mu dt)$, the extent of the bias is given by:[3]

$$E(R_A^N) = (1 + \mu dt)^N \left[1 + \frac{N(N-1)}{2} (1 + \mu dt)^{-2} \sigma_\xi^2 dt\right] + O(dt^2) \tag{9}$$

[2]Specifically, in Fama and French (1988a), p(t), the natural log of a stock price at time t, is the sum of a random walk, q(t), and a stationary component, z(t):

$$p(t) = q(t) + z(t) \text{ and } q(t) = q(t-1) + \mu + \varepsilon(t) \tag{3a}$$

where μ is expected drift and $\varepsilon(t)$ is white noise. z(t) follows a first-order autoregression (AR1) process:

$$z(t) = \phi z(t-1) + \eta(t) \tag{3b}$$

where $\eta(t)$ is white noise and ϕ is less than 1. From Equations (3a) and (3b), we compute a continuously compounded return:

$$p(t) - p(t-1) = [q(t) - q(t-1)] + [z(t) - z(t-1)] = \mu + \varepsilon(t) + \eta(t) + (\phi-1)z(t-1) \tag{3c}$$

Through successive substitutions for $z(\cdot)$ from Equations (3b) into (3c), the consistency between our formulation and that of Fama and French (1988a) follows from a comparison of Equations (3c) and (3).

[3]Derivations of the extent of biases in the arithmetic and geometric averages are available from the authors on request.

where $O(dt^2)$ denotes an order of no greater than dt^2, $\lim O(dt^2) \to 0$ as $dt \to 0$. From Equation (5), $\xi\sqrt{dt} = T^{-1}\Sigma_{t=1}^{T} v_t\sqrt{dt}$, and

$$\begin{aligned}\sigma_\xi^2 dt &= E[(\xi\sqrt{dt})^2] = T^{-2}(T\sigma_\eta^2 dt + \Sigma_{i=1}^{T}(T-i)\rho^{2i}\sigma_\eta^2 dt)\\ &\quad + T^{-2}(T\sigma_\xi^2 dt) = T^{-1}(\sigma_\eta^2 dt + \sigma_\varepsilon^2 dt)\\ &\quad + T^{-1}\left(\frac{(T+1)}{2}\right)\rho^{2\tau}\sigma_\eta^2 dt\end{aligned} \tag{10}$$

since by the mean value theorem there exists a τ, $T > \tau > 1$ such that $\Sigma_{t=1}^{T}(T-i)\rho^{2i} = \sum_{t=1}^{T}(T-i)\rho^{2\tau}$.

For $\rho = 0$ and fixed N, it is clear that the estimator R_A^N is asymptotically unbiased and consistent as $T \to \infty$, but for a finite and small T, is upward-biased for $N > 1$ by an amount proportional to the number of periods, $[N(N-1)/2]$, and variance, $T^{-1}(\sigma_\eta^2 dt + \sigma_\varepsilon^2 dt)$. Furthermore, for $\rho < 0$ and fixed N, the estimator R_A^N is asymptotically unbiased and consistent only for $N = 1$. For $N > 1$, the amount of upward bias is proportional to the number of periods, $[N(N-1)/2]$, and either the variance $\frac{1}{2}\rho^{2\tau}\sigma_\eta^2 dt$ for $T \to \infty$, or the variance $T^{-1}(\sigma_\eta^2 dt + \sigma_\varepsilon^2 dt) + T^{-1}[(T+1)/2]\rho^{2\tau}\sigma_\eta^2 dt$ for a finite and small T. Compounding the single-period arithmetic return tends to produce an estimated long-run return, and thus a risk premium, that is too high relative to the true mean $(1 + \mu dt)^N$.

C. The Bias in the Geometric Average

From a sample of T observations, the geometric average, R_G, is computed as:

$$R_G = \left(\Pi_{t=1}^{T} R_t\right)^{1/T} \tag{11}$$

and the estimated N-period return, R_G^N, as:

$$R_G^N = \left(\Pi_{t=1}^{T} R_t\right)^{N/T} = \exp\left\{\frac{N}{T}\Sigma_{t=1}^{T}\ln R_t\right\} \tag{12}$$

Hence, for a fixed N and $T \to \infty$, it is clear from Equation (12) that

$$\begin{aligned}p\lim R_G^N &= \exp\left\{p\lim \frac{N}{T}\Sigma_{t=1}^{T}\ln R_t\right\} = \exp\{NE[\ln R_t]\}\\ &< \exp\{N\ln[E(R_t)]\} < 1 + \mu dt)\end{aligned} \tag{13}$$

The geometric average is asymptotically biased downwards and thus is an inconsistent estimator of the long-run expected return.

To examine the bias for a fixed N and finite T, we rewrite the geometric average as:

$$\begin{aligned}R_G^N &= \left(\Pi_{t=1}^{T} R_t\right)^{N/T} = \Pi_{t=1}^{T}(1 + \mu dt + v_t\sqrt{dt})^{N/T}\\ &= [(1 + \mu dt)^T + \zeta\sqrt{dt}]^{N/T}\end{aligned} \tag{14}$$

where

$$\zeta\sqrt{dt} = \Pi_{t=1}^{T}(1 + \mu dt + v_t\sqrt{dt}) - (1 + \mu dt)^T \tag{15}$$

Taking the expectation of Equation (14) and a Taylor series expansion around $(1 + \mu dt)^T$ yields:

$$\begin{aligned}E(R_G^N) &= E[(1 + \mu dt)^T + \zeta\sqrt{dt}]^{N/T} = (1 + \mu dt)^N\\ &\quad + \left(\frac{N}{T}\right)(1 + \mu dt)^{N-T}E(\zeta\sqrt{dt}) + \left(\frac{N}{T}\right)\left(\frac{N}{T} - 1\right)\\ &\quad (1 + \mu dt)^{N-2T}E(\zeta\sqrt{dt})^2 + O(dt^2)\end{aligned} \tag{16}$$

where

$$E(\zeta\sqrt{dt}) = (1 + \mu dt)^{T-2}[\Sigma_{i=1}^{T-1}\rho^{2i-1}\Sigma_{j=1}^{T-i} j\rho^{T-i-j}]\sigma_\eta^2 dt + O(dt^2) \tag{17}$$

and

$$\begin{aligned}E(\zeta\sqrt{dt})^2 &= (1 + \mu dt)^{2(T-1)}[T(\sigma_\varepsilon^2 dt + \sigma_\eta^2 dt) + \rho^2\sigma_\eta^2 dt\\ &\quad \Sigma_{i=1}^{T-1}(T-i)\rho^i + 2\sigma_\eta^2 dt\,\Sigma_{i=1}^{T-1}\rho^{2i-1}\Sigma_{j=1}^{T-i} j\rho^{T-i-j}]\\ &\quad + O(dt^2)\end{aligned} \tag{18}$$

Observe that for $\rho=0$,

$$E(R_G^N) = (1 + \mu dt)^N\{1 + (1 + \mu dt)^{-2}\left(\frac{N}{T} - 1\right)[T(\sigma_\varepsilon^2 dt + \sigma_\eta^2 dt)]\} \tag{19}$$

the geometric average is downward-biased for $N < T$ but unbiased as $N \to T$. For $\rho < 0$,

$$\begin{aligned}E(R_G^N) &= (1 + \mu dt)^N\{1 + \left(\frac{N}{T}\right)(1 + \mu dt)^{-2}[E(\zeta\sqrt{dt})\\ &\quad + \left(\frac{N}{T} - 1\right)E(\zeta\sqrt{dt})^2]\}\end{aligned} \tag{20}$$

By definition, $E(\zeta\sqrt{dt})^2 = Var(\zeta\sqrt{dt}) > 0$, and it can be shown that $E(\zeta\sqrt{dt}) \le 0$ for $\rho \le 0$.[4] Hence, from Equation (20), the geometric average is always biased downward for $\rho < 0$, even as $N \to T$. It is also clear from Equation (20) that an increase in the stationary variance $\sigma_\varepsilon^2 dt$ raises the magnitude of the downward bias. The effect on the bias of changes in the parameters governing the temporal variation in expected returns, namely, ρ and $\sigma_\eta^2 dt$, is generally ambiguous. However, when $N \to T$,

$$E(R_G^N) = (1 + \mu dt)^N\{1 + (1 + \mu dt)^{-2}[1 + (T-2)\rho]\rho\sigma_\eta^2 dt + O(\rho^3)\sigma_\eta^2 dt\} \tag{21}$$

the downward bias at the limit is an increasing function of ρ and $\sigma_\eta^2 dt$.

[4]The sketch of the proof is as follows. Let T = 5. Compute and sum the five variances and ten covariances of $v_t\sqrt{dt}$. Examining the covariance sum for $\rho \le 0$ results in $E(\zeta\sqrt{dt}) \le 0$. The general result is obtained by induction. The formal derivation is available from the authors on request.

II. Simulation Results

We use simulations to assess the severity of the biases in the arithmetic and geometric averages. In addition, we present two other estimates of expected return, as suggested in Blume (1974): a weighted average and an overlapping average.

We calculate the weighted average as a horizon-weighted average of the arithmetic and geometric averages:

$$E(W^N) = \frac{T - N}{T - 1} R_A^N + \frac{N - 1}{T - 1} R_G^N \tag{22}$$

where the weights sum to one. When N=1, the arithmetic average receives all the weight. As $N \rightarrow T$, more weight is given to the geometric average.

We construct the overlapping average as follows. We compute an N-period total return, T-N+1 in number, by multiplying the first through the N^{th} one-period total returns together, the second through the $(N+1)^{st}$ one-period returns together, and so on. We then average the overlapped total returns.

To examine the empirical properties of each estimator, we use the return generating process described in Equation (3). For a benchmark monthly return, $\mu = 0.01$, and alternative values of autocorrelations $\rho = 0, -0.05, -0.25$, we draw a total of 250,000 random values of $\varepsilon_t\sqrt{dt}$ and $\eta_t\sqrt{dt}$ from zero mean normal variates with variances ranging from zero to 0.0081 for σ_ε^2 and zero to 0.0045 for σ_η^2, respectively. We then partition the 250,000 returns into 1,000 samples of 250 observations (T =250), and calculate the values of the four estimators for horizons N = 12,24,60,84,120.

Table 1 presents the simulation results when the autocorrelation and time-varying variance components are absent, i.e., $\rho = 0$ and $\sigma_\eta^2 = 0$. Simulation results in the presence of both time-varying and stationary variance as well as negative autocorrelation components appear in Table 2 (ρ =-0.05) and Table 3 (ρ = -0.25).

For the four estimators, the patterns of bias (direction and magnitude) and efficiency (standard deviation or the 0.05-0.95 fractile values) that appear in Table 1 are similar to those found in Blume (1974). Notice from Table 1 that for any investment horizon and stationary variance, the geometric average is always biased downward. For longer horizons N (=60,84,120), the arithmetic average is upward-biased, regardless of the stationary variance. For shorter horizons, N (=12,24), the arithmetic average is downward-biased for a small value of stationary variance, σ_ε^2 (= 0.0036), but upward-biased for a large value of stationary variance, σ_ε^2 (= 0.0081). For a small value of stationary variance, σ_ε^2 (= 0.0036), the overlapping estimator is downward-biased for any horizon, but for a large value of stationary variance, σ_ε^2 (= 0.0081), the estimator is upward-biased for shorter horizons, N (=12,24), and downward-biased for longer horizons, N (=60,84,120). Finally, for any horizon, the weighted average estimator is downward-biased for a small value of stationary variance, σ_ε^2 (= 0.0036),and upward-biased for a large value of stationary variance, σ_ε^2 (= 0.0081).

The magnitude of the bias is the largest for the geometric average. In addition, observe that for the smaller value of stationary variance, σ_ε^2 (= 0.0036), the arithmetic average has the least bias for shorter horizons, N (= 12,24), and the overlapping average the least bias for longer horizons, N (= 60,84,120). For the large value of stationary variance, σ_ε^2 (= 0.0081), and any horizon, the weighted and overlapping averages have less bias than the arithmetic and geometric averages. Overall, the geometric average is the most efficient estimator, and the overlapping average is the least efficient. The weighted average is consistently more efficient than the arithmetic and overlapping averages.

If we compare both Panel A's in Tables 1 and 2, we see that the arithmetic and geometric averages are more upward- and less downward-biased, respectively, and that both averages are less efficient. This represents the combined effect of a small negative autocorrelation (ρ = -0.05) and time-varying variance (σ_ε^2 = 0.0036),which is greater than that of σ_ε^2 alone. Moreover, although the bias for all estimators increases with N, the weighted average is not only the least biased, but is also more efficient than the overlapping average.

Similarly, if we compare Panels A and B of Table 2, introducing σ_ε^2 (= 0.0045) to a small negative autocorrelation (ρ = -0.05) and time-varying variance (σ_ε^2 = 0.0036) magnifies the magnitude of bias for all estimators. The overlapping average is the least biased, but least efficient, estimator. The weighted average is only slightly more biased, but is more efficient than the overlapping average.

Finally, the relative impact of σ_ε^2 and σ_η^2 is evident when we compare Panels B and C of Table 2. When $\sigma_\eta^2 > \sigma_\varepsilon^2$, the weighted average contains consistently smaller biases than when $\sigma_\eta^2 < \sigma_\varepsilon^2$, and its efficiency improves as N increases. Although the overlapping average is still the least biased, it is also the least efficient estimator. The weighted average is only slightly more biased, but is more efficient, than the overlapping average.

In general, the direction and magnitude of the biases reported in Table 2 are also observed in Table 3. In the majority of the cases reported in Table 3, however, the weighted average is the least biased of all estimators, although this improvement is achieved at the expense of efficiency. If we compare Panels A and C, we also

Table 1. Simulation Results in the Absence of Autocorrelation and Time-Varying Variance, $\rho = 0$ and $\sigma_\eta^2 = 0$

Monthly benchmark return is 1%. Horizon is stated in the number of months. Wt. Ave. is the horizon-weighted average of the arithmetic and geometric averages. Overlap is the overlapping average.

Panel A. $\rho = 0,\ \sigma_\eta^2 = 0, \sigma_\varepsilon^2 = 0.0036$

					Fractiles		
Estimator	Horizon	Benchmk Return	Average	Standard Error	0.05	0.50	0.95
Arithmetic	12	1.1268	1.1254	0.0507	1.0427	1.1246	1.2076
Geometric			1.1018	0.0499	1.0209	1.1013	1.1831
Wt. Ave.			1.1243	0.0507	1.0417	1.1237	1.2064
Overlap			1.1251	0.0516	1.0427	1.1248	1.2090
Arithmetic	24	1.2697	1.2691	0.1146	1.0872	1.2648	1.4582
Geometric			1.2165	0.1104	1.0422	1.2128	1.3998
Wt. Ave.			1.2640	0.1142	1.0831	1.2604	1.4526
Overlap			1.2657	0.1191	1.0786	1.2610	1.4682
Arithmetic	60	1.8167	1.8422	0.4198	1.2325	1.7990	2.5677
Geometric			1.6575	0.3796	1.1088	1.6198	2.3181
Wt. Ave.			1.7966	0.4098	1.2036	1.7567	2.5050
Overlap			1.8022	0.4725	1.1562	1.7383	2.6531
Arithmetic	84	2.3067	2.3858	0.7693	1.3400	2.2752	3.7442
Geometric			2.0580	0.6672	1.1556	1.9645	3.2448
Wt. Ave.			2.2719	0.7337	1.2796	2.1701	3.5650
Overlap			2.2851	0.8909	1.1991	2.1236	3.9425
Arithmetic	120	3.3004	3.5698	1.6822	1.5190	3.2362	6.5931
Geometric			2.8912	1.3714	1.2295	2.6239	5.3736
Wt. Ave.			3.2319	1.5270	1.3830	2.9328	5.9712
Overlap			3.2528	1.9440	1.2160	2.7965	6.8591

Panel B. $\rho = 0,\ \sigma_\eta^2 = 0, \sigma_\varepsilon^2 = 0.0081$

					Fractiles		
Estimator	Horizon	Benchmk Return	Average	Standard Error	0.05	0.50	0.95
Arithmetic	12	1.1268	1.1306	0.0760	1.0079	1.1284	1.2583
Geometric			1.0774	0.0730	0.9599	1.0745	1.2022
Wt. Ave.			1.1281	0.0758	1.0059	1.1261	1.2556
Overlap			1.1283	0.0780	1.0047	1.1260	1.2605
Arithmetic	24	1.2697	1.2839	0.1727	1.0159	1.2734	1.5833
Geometric			1.1662	0.1581	0.9214	1.1544	1.4452
Wt. Ave.			1.2726	0.1713	1.0071	1.2624	1.5697
Overlap			1.2703	0.1791	0.9944	1.2607	1.5759
Arithmetic	60	1.8167	1.9316	0.6610	1.0403	1.8298	3.1544
Geometric			1.5195	0.5241	0.8149	1.4320	2.5107
Wt. Ave.			1.8299	0.6269	0.9857	1.7356	2.9926
Overlap			1.8074	0.6846	0.8913	1.6954	3.1078
Arithmetic	84	2.3067	2.5929	1.2706	1.0569	2.3301	4.9944
Geometric			1.8540	0.9167	0.7508	1.6531	3.6284
Wt. Ave.			2.3363	1.1471	0.9532	2.1020	4.5182
Overlap			2.2787	1.2826	0.7824	2.0096	4.7529
Arithmetic	120	3.3004	4.1676	3.0671	1.0823	3.3482	9.9503
Geometric			2.5834	1.9241	0.6640	2.0506	6.3036
Wt. Ave.			3.3788	2.4961	0.8798	2.7156	8.1821
Overlap			3.2201	2.7834	0.6314	2.4351	8.7221

Table 2. Simulation Results with a Small Autocorrelation ρ = -0.05

Monthly benchmark return is 1%. Horizon is stated in the number of months. Wt. Ave. is the horizon-weighted average of the arithmetic and geometric averages. Overlap is the overlapping average.

Panel A. $\rho = -0.05,\ \sigma^2_\eta = 0.036\ \sigma^2_\varepsilon = 0$

					Fractiles		
Estimator	**Horizon**	**Benchmk Return**	**Average**	**Standard Error**	**0.05**	**0.50**	**0.95**
Arithmetic	12	1.1268	1.1269	0.0515	1.0446	1.1237	1.2166
Geometric			1.1032	0.0506	1.0246	1.1003	1.1917
Wt. Ave.			1.1258	0.0515	1.0437	1.1226	1.2156
Overlap			1.1236	0.0527	1.0383	1.1221	1.2165
Arithmetic	24	1.2697	1.2724	0.1171	1.0913	1.2627	1.4801
Geometric			1.2195	0.1125	1.0499	1.2107	1.4201
Wt. Ave.			1.2674	0.1167	1.0872	1.2574	1.4748
Overlap			1.2621	0.1216	1.0743	1.2546	1.4707
Arithmetic	60	1.8167	1.8556	0.4393	1.2440	1.7918	2.6651
Geometric			1.6687	0.3962	1.1294	1.6127	2.4032
Wt. Ave.			1.8095	0.4286	1.2159	1.7476	2.6018
Overlap			1.7869	0.4676	1.1393	1.7179	2.6344
Arithmetic	84	2.3067	2.4123	0.8214	1.3575	2.2626	3.9446
Geometric			2.0793	0.7102	1.1858	1.9524	3.4127
Wt. Ave.			2.2966	0.7826	1.2986	2.1572	3.7665
Overlap			2.2608	0.8839	1.1510	2.1064	4.0036
Arithmetic	120	3.3004	3.6361	1.8669	1.5475	3.2106	7.1027
Geometric			2.9415	1.5153	1.2756	2.6007	5.7753
Wt. Ave.			3.2902	1.6915	1.4119	2.9204	6.4632
Overlap			3.2330	1.9575	1.1754	2.7698	6.8499

Panel B. $\rho = -0.05,\ \sigma^2_\eta = 0.036,\ \sigma^2_\varepsilon = 0.0045$

					Fractiles		
Estimator	**Horizon**	**Benchmk Return**	**Average**	**Standard Error**	**0.05**	**0.50**	**0.95**
Arithmetic	12	1.1268	1.1319	0.0748	1.0164	1.1283	1.2568
Geometric			1.0786	0.0720	0.9662	1.0763	1.1971
Wt. Ave.			1.1294	0.0747	1.0143	1.1259	1.2544
Overlap			1.1278	0.0771	1.0077	1.1238	1.2610
Arithmetic	24	1.2697	1.2867	0.1713	1.0331	1.2732	1.5796
Geometric			1.1686	0.1571	0.9335	1.1585	1.4330
Wt. Ave.			1.2754	0.1669	1.0239	1.2617	1.5668
Overlap			1.2720	0.1819	1.0056	1.2590	1.6056
Arithmetic	60	1.8167	1.9412	0.6685	1.0847	1.8290	3.1359
Geometric			1.5266	0.5307	0.8419	1.4446	2.4583
Wt. Ave.			1.8388	0.6343	1.0243	1.7300	2.9745
Overlap			1.8159	0.7385	0.9271	1.6760	3.1844
Arithmetic	84	2.3067	2.6111	1.3023	1.1206	2.3285	4.9536
Geometric			1.8663	0.9401	0.7859	1.6736	3.5227
Wt. Ave.			2.3524	1.1760	1.0025	2.0926	4.4684
Overlap			2.3005	1.4391	0.8698	1.9396	4.7906
Arithmetic	120	3.3004	4.2146	3.2132	1.1767	3.3451	9.8342
Geometric			2.6119	2.0128	0.7088	2.0869	6.0431
Wt. Ave.			3.4166	2.6141	0.9468	2.6988	7.9694
Overlap			3.3191	3.4287	0.7108	2.3538	8.5702

Table 2. Simulation Results with a Small Autocorrelation ρ = -0.05 (*Continued*)

Panel C. $\rho = -0.05$, $\sigma_\eta^2 = 0.0045$ $\sigma_\varepsilon^2 = 0.0036$

Estimator	Horizon	Benchmk Return	Average	Standard Error	Fractiles 0.05	0.50	0.95
Arithmetic	12	1.1268	1.1306	0.0749	1.0085	1.1289	1.2550
Geometric			1.0779	0.0720	0.9603	1.0771	1.1963
Wt. Ave.			1.1282	0.0747	1.0064	1.1265	1.2522
Overlap			1.1266	0.0779	0.9985	1.1242	1.2583
Arithmetic	24	1.2697	1.2839	0.1701	1.0172	1.2744	1.5750
Geometric			1.1670	0.1559	0.9223	1.1602	1.4312
Wt. Ave.			1.2727	0.1687	1.0084	1.2632	1.5609
Overlap			1.2689	0.1828	0.9850	1.2568	1.5954
Arithmetic	60	1.8167	1.9297	0.6472	1.0435	1.8333	3.1133
Geometric			1.5206	0.5141	0.8168	1.4500	2.4503
Wt. Ave.			1.8287	0.6141	0.9896	1.7368	2.9461
Overlap			1.8123	0.7192	0.8688	1.6657	3.1331
Arithmetic	84	2.3067	2.5865	1.2395	1.0614	2.3363	4.9036
Geometric			1.8538	0.8962	0.7533	1.6824	3.5067
Wt. Ave.			2.3320	1.1197	0.9580	2.1085	4.4085
Overlap			2.2913	1.3224	0.7811	1.9445	4.7278
Arithmetic	120	3.3004	4.1422	2.9827	1.0888	3.3611	9.6930
Geometric			2.5764	1.8779	0.6672	2.1025	6.0039
Wt. Ave.			3.3626	2.4308	0.8854	2.7379	7.8210
Overlap			3.2489	2.8583	0.6348	2.3838	8.1933

Table 3. Simulation Results with a Large Autocorrelation ρ = -0.25

Monthly benchmark return is 1%. Horizon is stated in the number of months. Wt. Ave. is the horizon-weighted average of the arithmetic and geometric averages. Overlap is the overlapping average.

Panel A. $\rho = -0.25$, $\sigma_\eta^2 = 0.00108$ $\sigma_\varepsilon^2 = 0.00252$

Estimator	Horizon	Benchmk Return	Average	Standard Error	Fractiles 0.05	0.50	0.95
Arithmetic	12	1.1268	1.1262	0.0487	1.0448	1.1266	1.2077
Geometric			1.1021	0.0478	1.0213	1.1024	1.1816
Wt. Ave.			1.1251	0.0486	1.0437	1.1254	1.2065
Overlap			1.1225	0.0494	1.0386	1.1221	1.2011
Arithmetic	24	1.2697	1.2708	0.1097	1.0915	1.2692	1.4585
Geometric			1.2169	0.1054	1.0431	1.2152	1.3962
Wt. Ave.			1.2656	0.1092	1.0869	1.2638	1.4527
Overlap			1.2603	0.1136	1.0728	1.2567	1.4536
Arithmetic	60	1.8167	1.8458	0.3996	1.2447	1.8149	2.5689
Geometric			1.6565	0.3602	1.1113	1.6280	2.3034
Wt. Ave.			1.7991	0.3898	1.2134	1.7704	2.5056
Overlap			1.7895	0.4342	1.1623	1.7311	2.5611
Arithmetic	84	2.3067	2.3891	0.7302	1.3586	2.3035	3.7467
Geometric			2.0536	0.6308	1.1592	1.9784	3.2159
Wt. Ave.			2.2726	0.6955	1.2935	2.1953	3.5686
Overlap			2.2606	0.7989	1.1846	2.1236	3.7313
Arithmetic	120	3.3004	3.5665	1.5918	1.5493	3.2937	6.5994
Geometric			2.8738	1.2908	1.2349	2.6504	5.3055
Wt. Ave.			3.2216	1.4415	1.3994	2.9794	5.9669
Overlap			3.2091	1.6643	1.1889	2.8265	6.4095

Table 3. Simulation Results with a Large Autocorrelation ρ = -0.25 (*Continued*)

Panel B. $\rho = -0.25$, $\sigma^2_\eta = 0.000405$ $\sigma^2_\varepsilon = 0.007695$

Estimator	Horizon	Benchmk Return	Average	Standard Error	Fractiles 0.05	0.50	0.95
Arithmetic	12	1.1268	1.1299	0.0785	1.0006	1.1268	1.2676
Geometric			1.0768	0.0756	0.9512	1.0737	1.2076
Wt. Ave.			1.1275	0.0783	0.9980	1.1244	1.2646
Overlap			1.1264	0.0812	0.9936	1.1230	1.2652
Arithmetic	24	1.2697	1.2829	0.1789	1.0011	1.2696	1.6069
Geometric			1.1652	0.1643	0.9049	1.1528	1.4583
Wt. Ave.			1.2715	0.1775	0.9908	1.2584	1.5910
Overlap			1.2679	0.1898	0.9755	1.2511	1.5983
Arithmetic	60	1.8167	1.9326	0.6969	1.0028	1.8162	3.2732
Geometric			1.5208	0.5546	0.7788	1.4267	2.5679
Wt. Ave.			1.8309	0.6615	0.9445	1.7202	3.0817
Overlap			1.8186	0.7458	0.8661	1.6569	3.2862
Arithmetic	84	2.3067	2.6022	1.3673	1.0040	2.3058	5.2596
Geometric			1.8619	0.9902	0.7047	1.6447	3.7447
Wt. Ave.			2.3451	1.2358	0.8964	2.0758	4.6840
Overlap			2.3242	1.4276	0.7842	1.9571	5.1075
Arithmetic	120	3.3004	4.2200	3.4602	1.0057	3.2985	10.7135
Geometric			2.6200	2.1793	0.6066	2.0356	6.5943
Wt. Ave.			3.4233	2.8210	0.8030	2.6675	8.5390
Overlap			3.3601	3.1676	0.6356	2.3754	9.7576

Panel C. $\rho = -0.25$, $\sigma^2_\eta = 0.00243$ $\sigma^2_\varepsilon = 0.00567$

Estimator	Horizon	Benchmk Return	Average	Standard Error	Fractiles 0.05	0.50	0.95
Arithmetic	12	1.1268	1.1294	0.0721	1.0199	1.1252	1.2561
Geometric			1.0753	0.0694	0.9690	1.0721	1.1970
Wt. Ave.			1.1269	0.0719	1.0174	1.1225	1.2533
Overlap			1.1200	0.0738	1.0113	1.1146	1.2504
Arithmetic	24	1.2697	1.2808	0.1641	1.0403	1.2661	1.5779
Geometric			1.1611	0.1505	0.9390	1.1493	1.4329
Wt. Ave.			1.2693	0.1628	1.0296	1.2543	1.5632
Overlap			1.2529	0.1700	1.0132	1.2368	1.5553
Arithmetic	60	1.8167	1.9141	0.6252	1.1038	1.8038	3.1274
Geometric			1.4987	0.4957	0.8545	1.4161	2.4576
Wt. Ave.			1.8115	0.5930	1.0404	1.7044	2.9563
Overlap			1.7524	0.6358	0.9180	1.6407	2.9633
Arithmetic	84	2.3067	2.5532	1.1906	1.1483	2.2839	4.9347
Geometric			1.8140	0.8578	0.8024	1.6276	3.5213
Wt. Ave.			2.2965	1.0745	1.0309	2.0482	4.4316
Overlap			2.1744	1.1431	0.8366	1.9151	4.4332
Arithmetic	120	3.3004	4.0541	2.8088	1.2184	3.2539	9.7808
Geometric			2.4915	1.7562	0.7301	2.0054	6.0396
Wt. Ave.			3.2761	2.2832	0.9765	2.6212	7.8862
Overlap			2.9808	2.3220	0.6750	2.2822	7.5861

Table 3. Simulation Results with a Large Autocorrelation ρ = -0.25 (*Continued*)

Panel D. $\rho = -0.25$, $\sigma_{\eta}^{2} = 0.0036$ $\sigma_{\varepsilon}^{2} = 0.0045$

Estimator	Horizon	Benchmk Return	Average	Standard Error	Fractiles 0.05	Fractiles 0.50	Fractiles 0.95
Arithmetic	12	1.1268	1.1275	0.0709	1.0146	1.1272	1.2492
Geometric			1.0730	0.0684	0.9633	1.0725	1.1877
Wt. Ave.			1.1250	0.0708	1.0125	1.1247	1.2467
Overlap			1.1158	0.0724	1.0008	1.1168	1.2410
Arithmetic	24	1.2697	1.2762	0.1605	1.0295	1.2705	1.5606
Geometric			1.1560	0.1474	0.9280	1.1503	1.4107
Wt. Ave.			1.2646	0.1592	1.0207	1.2593	1.5468
Overlap			1.2446	0.1662	0.9894	1.2401	1.5459
Arithmetic	60	1.8167	1.8947	0.6019	1.0754	1.8196	3.0423
Geometric			1.4809	0.4767	0.8296	1.4190	2.3638
Wt. Ave.			1.7925	0.5707	1.0183	1.7202	2.8760
Overlap			1.7249	0.6193	0.8986	1.6286	2.9045
Arithmetic	84	2.3067	2.5137	1.1352	1.1072	2.3119	4.7477
Geometric			1.7816	0.8146	0.7699	1.6323	3.3347
Wt. Ave.			2.2595	1.0233	0.9959	2.0773	4.2567
Overlap			2.1478	1.1423	0.8072	1.8783	4.4142
Arithmetic	120	3.3004	3.9518	2.6400	1.1565	3.3109	9.2557
Geometric			2.4201	1.6346	0.6883	2.0137	5.5876
Wt. Ave.			3.1891	2.1377	0.9301	2.6705	7.4157
Overlap			2.9632	2.3759	0.6444	2.2599	7.7379

observe that when σ_{ε}^{2} and σ_{η}^{2} both increase by the same proportion, the weighted average experiences a smaller bias relative to the other three estimators. Furthermore, we see from Panels B and C that a reduction in σ_{ε}^{2} that is offset by a corresponding increase in σ_{η}^{2} improves the weighted average's efficiency.

The effect of higher negative autocorrelation is evident when we compare Panel D in Table 3 with Panel B in Table 2. Even though we obtain a higher efficiency for all estimators, a higher negative autocorrelation ρ leads to a smaller bias in the arithmetic and weighted averages, but a larger bias for the geometric and overlapping averages. Moreover, although Table 3 shows that the weighted average is the second most efficient estimator, it is overall the least biased when negative autocorrelation, time-varying, and stationary variance components are all present.

III. Concluding Remarks

We show that both the arithmetic and geometric averages are biased estimates of long-run expected returns, and the bias increases with the length of the investment horizons. The existence of negative autocorrelation in long-horizon returns documented by Fama and French (1988a, 1988b), Lo and MacKinlay (1988), and Poterba and Summers (1988) exacerbates the bias. The implication is that without making an adjustment, we are likely to obtain an estimate of long-run expected return (and risk premium) that is either too high or too low, and this can result in an inappropriate decision to reject a good project or accept a bad project.

The horizon-weighted average of the arithmetic and geometric averages, proposed by Blume (1974), is an alternative estimate of long-run expected returns. Our simulation results indicate that in general, the horizon-weighted average contains the least bias. It is also more efficient than other estimators in the presence of negative autocorrelation, time-varying, and stationary variances. This conclusion contrasts with Blume's conjecture that "...if one cannot assume independence of successive one-period relatives or if there is even a slight chance that these relatives are dependent, the simple average of N-period relatives would appear preferable to the nonlinear estimators which, even under ideal conditions, yield only a modest increase in efficiency." ■

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