A Reconsideration of the Equity Premium Puzzle

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Abstract

This paper develops a tractable asset pricing framework based on an Arrow Debreu economy with heterogeneous agents. In this setting, market payoffs, not aggregate consumption, is the appropriate covariate to price assets. To test the model, I use data of several developed economies from Campbell (2003, 2017) to find a median value of relative risk aversion of 1.57 and a time preference rate of 4.58%. The paper develops a formula for expected individual security returns similar to the CAPM.

Code words: Asset pricing, complete markets, equity risk premium puzzle, risk free rate puzzle. JEL D53, E10, E21, G12, G13, G30, G32.

1 Introduction

Consumption based asset pricing (CBAP), pioneered by Rubinstein (1976) and Breeden (1979), is an extensively used framework in finance and macroeconomics. It has two appealing features: the first is its dynamic focus. The

*I would like to thank Prof. John Campbell for sharing data on aggregate dividends. All errors are of course my responsibility

second is its use of a representative agent framework, which allows economists to study and measure variables of great interest, such as the risk aversion and impatience of agents. In this regard, Mehra and Prescott (1985) found that in a CBAP framework, the relative risk aversion parameters needed to justify the observed equity risk premium were unreasonably large. Weil (1989) found a similar result about the time preference rates consistent with observed risk free yields. A large literature emerged to verify and explain the equity premium and risk free rate puzzles. One point of consensus in this area is that aggregate consumption is too smooth to explain the observed returns. The interested reader is referred to Campbell (2003, 2017) for an in depth explanation of the different models involved in that research program.

Consumption based asset pricing has rarely been used to predict expected returns of individual securities. Instead, that field of empirical asset pricing has looked for factors that affect the cross section of returns. Harvey et al. (2016) provide an extensive recount of the hundreds of articles written on this topic.

This paper develops a tractable framework based on an Arrow Debreu economy with heterogeneous agent types. Why focus on heterogeneity? Wolff (2014) has documented that about half of U.S. households have no direct or indirect stock holdings, and that the top 10% of households own about 80% of outstanding stocks[1] . These findings suggest that a representative agent model may not be a useful simplifying assumption. Intuitively, it seems that the consumption of the 90% of households with very little stock has almost no impact on asset prices. To operationalize this intuition, I develop a model that has agents with tradable endowments, and agents who live autarkically. The framework allows for investor aggregation, and supposes log-normally distributed total endowments. With these assumptions, I reproduce some theoretical results found in the literature, namely a formula for the risk free rate. This model also yields a simple formula for the expected market return and equity premium.

I use data from Campbell (2003, 2017) to recalculate the relative risk aver-

[1] Wolff (2014), Table 2b and Table 3a

sions and time preference rates for several economies. I find consistent and reasonable results, with a median relative risk aversion parameter of 1.57 and a median time preference rate of 4.58%. These estimates are much closer to the results found in experimental data by Holt and Laury (2002), Andersen et al. (2008), and studies using option prices by Bliss and Panigirtzoglou (2004), suggesting that the results from small stake experiments do not in practice blow out of proportion, as argued by Rabin (2000).

The pricing framework here can also be used to value individual securities. I find that for a class of cash flows, one can reproduce a non-linear version of the CAPM. In this result, it becomes clear that the appropriate stochastic discount factor are the market payoffs, modulated by the risk aversion and impatience of investors. The results of this model combine the original CAPM emphasis on the market portfolio with the CBAP analysis of investor preferences (risk aversion and impatience) to value assets.

Surprisingly, the extension of the model to a multi-period framework is a very simple one. With some assumptions about the behavior of endowment growth, one can develop a simple yield curve.

This article is structured as follows: section 2 develops the complete markets model. Section 3 derives the formulas for the risk free rate and the expected market return. These formulas are used to estimate the relative risk aversion and time preference rate parameters. The fourth section develops a pricing framework for individual securities, and looks at a special class of cash flows. In that framework, a modified version of the CAPM emerges. The fifth section extends the model in a multi period framework, and the sixth section concludes.

2 The Model

The model uses a complete markets framework developed by Arrow (1964) and Debreu (1959). The specific assumptions here are:

1. An exchange economy with exogenous individual endowments.

2. The economy has two periods, $t = 0, 1$, with uncertainty in period $t = 1$. This is extended to a multi-period framework in the last section of this article.

3. In period $t = 1$ there are a number states of nature, that can be either finite or continuous. The set of states of nature is S. Any given state of nature is called $s \epsilon S$. Agents have homogeneous beliefs about the probability of a state s, defined as π_s. For discrete states, π_s is a simple probability, while if there are continuous states, π_s is a density function. The price of an Arrow-Debreu security is p_s for $s \epsilon S$. I will define everything in terms of the goods at time $t = 0$, that is $p_0 \equiv 1$.

4. There are I investors with separable cardinal utilities, so for $i = 1, 2..., I$:

$$U_i = u_i(c_{i0}) + e^{-\delta_i} u_i(c_{it})$$

$$u_i(c_{it}) = \frac{c_{it}^{1-\gamma_i} - 1}{1 - \gamma_i} \rightarrow u_i'(c_{it}) = c_{it}^{-\gamma_i}$$

   The time preference rate for each investor is δ_i , and their relative risk aversion is γ_i. Each investor i has tradable endowments e_{i0} and e_{is} $\forall s \epsilon S$. Aggregate investor endowments are $E_0 \equiv \sum_{i=1}^{I} e_{i0}$ and $E_s \equiv \sum_{i=1}^{I} e_{is}$.

5. There are another K autarkic agents who receive wages w_{it} in both periods for $i = I + 1, ...I + K$. The fixed aggregate value of these wages is $W \equiv \sum_{i=I+1}^{I+K} w_{i0} = \sum_{i=I+1}^{I+K} w_{i1}$. Aggregate consumption in this economy is $C_0 = E_0 + W$ and $C_s = E_s + W$.

6. There are J complex securities with a value V_j for $j = 1, ..., J$. A complex security with cash flows cf_{js} is valued by a no arbitrage condition, yielding $V_j = \sum_{s \epsilon S} p_s cf_{js}$ for finite states, and $V_j = \int_{s \epsilon S} p_s cf_{js} ds$ for continuous states.

The decentralized equilibrium is found by solving the individuals' maximization, that is:

$$L_i = u_i(c_{i0}) + e^{-\delta_i} \sum_{s \epsilon S} \pi_s u_i(c_{is}) ds + \lambda_i \left[(e_{i0} - c_{i0}) + \sum_{s \epsilon S} p_s \left(e_{is} - c_{is} \right) \right]$$

In the case of continuous states, one would use $\int_{s \epsilon S} ds$ instead of $\sum_{s \epsilon S}$. The first order conditions to this problem are:

$$\frac{\partial L_i}{\partial c_{i0}} = u_i'(c_{i0}) - \lambda_i = 0 \to \lambda_i = u_i'(c_{i0})$$

$$\frac{\partial L_i}{\partial c_{is}} = e^{-\delta_i} \pi_s u_i'(c_{is}) - \lambda_i p_s = 0 \to \lambda_i = \frac{e^{-\delta_i} \pi_s}{p_s} u_i'(c_{is}) = u_i'(c_{i0})$$

Using the power utilities, we get the following well known result:

$$p_s = e^{-\delta_i} \pi_s \left(\frac{c_{i0}}{c_{is}} \right)^{\gamma_i}$$

This equation tells us how individual consumption should adjust to state prices and probabilities. To estimate equilibrium prices and returns based on aggregate investor endowments, I assume that $\delta_i = \delta$ and $\gamma_i = \gamma$, for $i = 1, ..., I$.[2] With this simplification, I get partial aggregation as follows:

$$c_{i0} = \left(\frac{p_s}{e^{-\delta} \pi_s} \right)^{1/\gamma} c_{is}$$

$$E_0 = \sum_{i=1}^{I} c_{i0} = \left(\frac{p_s}{e^{-\delta} \pi_s} \right)^{1/\gamma} \sum_{i=1}^{I} c_{is} = \left(\frac{p_s}{e^{-\delta} \pi_s} \right)^{1/\gamma} E_s$$

The equilibrium state prices can be re-written as:

$$p_s = e^{-\delta} \pi_s \left(\frac{E_s}{E_0} \right)^{-\gamma} \tag{1}$$

[2]No such assumption is needed for agents $i = I + 1, ..I + K$.

State prices rise with a fall in δ and E_s. p_s go up with increases in π_s and E_0. For states of nature, that is $\frac{E_s}{E_0} < 1$, p_s increases as γ rises, while for abundant resources the opposite is true. If we assume that $\frac{E_s}{E_0}$ has a log-normal distribution, with $ln\left(\frac{E_s}{E_0}\right) \sim N(\mu, \sigma^2)$, the state prices become:

$$p_s = \pi_s exp\left(-\delta - \gamma\mu - \gamma\sigma s\right) = \frac{1}{\sqrt{2\pi}} exp\left(-\delta - \gamma\mu - \gamma\sigma s - \frac{s^2}{2}\right)$$

Where $s \sim N(0,1)$ and hence $\pi_s = \frac{1}{\sqrt{2\pi}} exp\left(-\frac{s^2}{2}\right)$. I will use equation (1) throughout, since it is the one that yields more economic intuition.

3 Prices and expected returns of two fundamental assets

3.1 The risk free asset

Consider an economy with continuous states. A risk free security is the one that offers one unit for every state of nature at $t = 1$, and has a value V_{r_f} given by:

$$V_{r_f} \equiv \int_{s\epsilon S} p_s ds = e^{-\delta} \int_{s\epsilon S} \pi_s \left(\frac{E_s}{E_0}\right)^{-\gamma} = e^{-\delta} E\left[\left(\frac{E_s}{E_0}\right)^{-\gamma}\right]$$

Consider a continuously compounded rate r_f, where $V_{r_f} \equiv e^{-r_f}$. If $ln\left(\frac{E_s}{E_0}\right) \sim N(\mu, \sigma^2)$, where μ growth rate of the logs of aggregate endowments, and σ^2 its variance, then $\left(\frac{E_s}{E_0}\right)^{-\gamma}$ is also log-normal, since $ln\left[\left(\frac{E_s}{E_0}\right)^{-\gamma}\right] = -\gamma ln\left[\left(\frac{E_s}{E_0}\right)\right] \sim N(-\gamma\mu, \gamma^2\sigma^2)$. The expected value of the log-normal variable is $E\left[\left(\frac{E_s}{E_0}\right)^{-\gamma}\right] = e^{-\gamma\mu + \frac{1}{2}\gamma^2\sigma^2}$. Putting all these results together we obtain:

$$V_{r_f} = e^{-r_f} = e^{-\delta - \gamma\mu + \frac{1}{2}\gamma^2\sigma^2}$$

Taking the natural logarithm and re-arranging terms, we obtain a formula for the continuously compounded risk free rate:

$$r_f = \delta + \gamma\mu - \frac{1}{2}\gamma^2\sigma^2 \tag{2}$$

This is identical to the formula developed in standard consumption based asset pricing, as in Campbell (2017). The continuously compounded risk free rate increases as the time preference rate δ rises, and as expected growth μ increases. The effect of expected growth on r_f is greater in economies with high risk aversion. Expected volatility lowers the risk free rate, as it depresses p_s generally. This effect is again stronger for economies with highly risk averse agents. Finally, risk aversion has an ambiguous effect on r_f, as $\frac{\partial r_f}{\partial \gamma} = \mu - \gamma\sigma^2$. It is positive for $\gamma < \frac{\mu}{\sigma^2}$ and negative for $\gamma > \frac{\mu}{\sigma^2}$. We will find with the data in section 3 that all of the economies studied there fall under the second case.

3.2 The market portfolio

The market consists of all the traded assets in this economy, with an aggregate output of $cf_{ms} = E_s$. Its value is given by:

$$V_m = \int_{s\epsilon S} E_s p_s = E_0 e^{-\delta} \int_{s\epsilon S} \pi_s \left(\frac{E_s}{E_0}\right)^{1-\gamma} = E_0 e^{-\delta} E\left[\left(\frac{E_s}{E_0}\right)^{1-\gamma}\right]$$

$$V_m = E_0 e^{(1-\gamma)\mu + \frac{1}{2}(1-\gamma)^2\sigma^2 - \delta} = E_0 exp\left(\mu - \gamma\sigma^2 + \frac{\sigma^2}{2} - r_f\right)$$

The last part of the equality exploits the definition of r_f found in equation (2). Consider the continuously compounded return r_{ms} :

$$r_{ms} \equiv ln(1 + \widetilde{r}_{ms}) = ln\left(\frac{E_s}{V_m}\right) = ln\left(\frac{E_s}{E_0}\right) - ln\left(\frac{V_m}{E_0}\right)$$

Using the above result about the value of the market portfolio V_m and using the log-normality of $\frac{E_s}{E_0}$ we have:

$$r_{ms} = \mu + \sigma s + \left(r_f - \mu + \gamma\sigma^2 - \frac{\sigma^2}{2} \right)$$

$$r_{ms} = r_f + \gamma\sigma^2 - \frac{\sigma^2}{2} + \sigma s = \delta + \gamma\mu - \frac{\sigma^2}{2}(\gamma - 1)^2 + \sigma s$$

$$Var(r_{ms}) \equiv \sigma_m^2 = \sigma^2$$

This last equation is key in understanding the difference between a consumption based asset pricing framework and this model, since the variance of the endowments is equal to the variance of the market returns. On the other hand, aggregate consumption growth in this model is:

$$\frac{C_s}{C_0} = \frac{E_s + W}{E_0 + W} = \alpha \left(\frac{E_s}{E_0} \right) + (1 - \alpha)W$$

Where $\alpha \equiv \frac{E_0}{E_0+W}$ is the share of total period 0 endowments invested in Arrow Debreu securities. It follows that $var\left(\frac{C_s}{C_0}\right) = \alpha^2 var\left(\frac{E_s}{E_0}\right) \ll var\left(\frac{E_s}{E_0}\right)$. If one wishes to use a proxy for the volatility of endowments, the closest in this model is the volatility of the log change of aggregate dividends[3]. The expected value of the continuously compounded market return is:

$$E(r_{ms}) = r_f + \gamma\sigma^2 - \frac{\sigma^2}{2} \tag{3}$$

To study the market return in other contexts, it is useful to define the simple market returns are $\widetilde{r}_{ms} \equiv e^{r_{ms}} - 1$, with an expected value of $E(\widetilde{r}_{ms}) = exp(r_f + \gamma\sigma^2) - 1$. We can also define a discount factor $\hat{r}_m$ as the rate at which we bring the expected market cash flows to its current value, i.e. $V_m = e^{-\hat{r}_m} E\left(cf_{ms}\right)$, which in this case would be $\hat{r}_m = ln(1 + E(\widetilde{r}_{ms})) = E(r_{ms}) + \frac{\sigma^2}{2}$.

As can be seen, the same factors that affect the risk free rate will also change the expected market return, with some subtle differences. The expected market return increases with a rise in the time preference rate δ of investors, with the expected growth rate of endowments μ. An increase in

[3]After the early 1980s, this became a worse proxy, as share buybacks became an important source of income for investors

relative risk aversion and volatility raises the expected simple market return $E(\widetilde{r}_{ms})$. Equation (3) can be re-written to estimate the relative risk aversion:

$$\gamma = \frac{E(r_m) - r_f}{\sigma^2} + \frac{1}{2} \tag{4}$$

We can further simplify this estimate as $\gamma = \frac{\hat{r}_m - r_f}{\sigma_m^2} = \frac{\hat{SR}_m}{\sigma_m}$, i.e., a continuously compounded version of the Sharpe Ratio divided by the market standard deviation. Compare equation (4) with the standard CBAP implementation:

$$\gamma_{CBAP} = \frac{E(r_m) - r_f + \frac{\sigma_m^2}{2}}{\sigma_{cm}}$$

The complete market framework takes the CBAP to this implacable conclusion: general equilibrium implies that for those agents who determine the price of securities, their consumption has in fact the property that $\sigma_{cm} = \sigma_m^2$.

3.3 Numerical estimation of the equity risk premium

I estimate various utility parameters using two proxies for aggregate endowment volatility: stock market volatility, denoted σ_m, and the variance of log dividend growth, called here σ_d.

Table 1 shows data from Campbell (2003, 2017), in addition to the average log dividend average growth and volatility[4]. I have also transcribed the volatility of aggregate log consumption growth σ_c. Table 2 presents the different estimates for γ and δ.

The first four columns of Table 2 calculate the relative risk aversion parameter γ. γ_m uses σ_m, while γ_d uses σ_d as a proxy for log aggregate endowment growth volatility σ. The third and fourth columns show the estimates of γ using the consumption based asset pricing (CBAP) framework as reported by Campbell (2003, 2017) for two cases: γ_{CBAP} where σ_{cm} is directly used, and

[4]Which I have updated with data kindly obtained from John Campbell. I calculated annual real dividends from this data, and taken log growth rates, calculating μ_d and σ_d from 1970 to 2011. The annual data is chosen because of strong seasonality in almost all the economies shown here. The data for μ_d and σ_d for the last three rows, comes from Campbell (2003).

Table 1: Source data of returns, volatilities, and growth rates

Country	Range	r_m	r_f	σ_m	σ_d	σ_c	μ_d
AUL	1970.1-2011.2	3.840	2.000	20.750	11.890	1.770	1.654
CAN	1970.1-2011.2	5.470	2.070	17.850	8.848	1.930	1.548
FR	1973.2-2011.2	7.060	2.080	23.100	12.561	1.800	2.081
GER	1978.4-2011.2	7.540	2.380	23.850	13.971	4.190	2.174
ITA	1971.2-2011.2	1.510	1.860	25.740	23.667	2.230	-1.184
JAP	1970.2-2011.2	2.700	1.030	21.410	11.053	2.920	-0.249
NTH	1977.2-2011.2	8.570	2.290	19.760	12.687	2.210	1.112
SWT	1982.2-2011.2	8.140	0.870	20.050	12.312	1.300	3.652
SWD	1920-1998	7.084	2.209	18.641	12.894	2.816	1.551
UK	1919-1998	7.713	1.255	22.170	7.824	2.886	1.990
USA	1891-1998	7.169	2.020	18.599	14.019	3.218	1.516
Median		7.084	2.020	20.750	12.561	2.230	1.551

Numbers stated in percentage points. Here $d \equiv ln\left(\frac{DIV_t}{DIV_{t-1}}\right)$ and $c \equiv ln\left(\frac{C_t}{C_{t-1}}\right)$. The variables are r_m: average of the log return for the market. r_f log return for a short term risk free asset, σ_m standard deviation of the log market return. σ_c standard deviation for the log growth in aggregate consumption. Source: Campbell (2003), Tables 1 and 2 and Campbell (2017), table 6.1. For the other variables, we have σ_d as the standard deviation for the log growth in annual aggregate dividends, μ_d average of the log growth in annual dividends. Source: author calculations from Campbell data for all countries except Sweden, UK, and USA, which come from Campbell (2003).

γ_{2CBAP} where the correlation between aggregate consumption and market returns is set to one, so $\sigma_{cm} = \sigma_m\sigma_c$. The second parameter of interest is the time preference rate δ, shown in the last three columns of table 2. δ would be the risk free rate under risk neutrality.

When I use market volatility as the proxy for log endowment volatility, I get positive, stable, and reasonable relative risk aversion parameters, ranging from 0.437 in Italy to 2.308 for Switzerland, with a median value of 1.567. When I use the volatility of the log change in aggregate dividends, the estimate for γ increases in all but one country (Italy), and the median value is 3.432. δ_m, which uses σ_m has a median value of 4.58%. Meanwhile, δ_d which uses σ_d, has a median value of 5.02%. I also calculate the effect of increasing risk aversion

Table 2: Parameter Estimates of Complete Markets Model vs. CBAP

	Variable	γ_m	γ_d	γ_{CPAB}	γ_{2CPAB}	δ_m	δ_d	δ_{2CBAP}
						%	%	%
Country	Period							
AUL	1970.1-2011.2	0.927	1.802	<0	10.890	2.32	1.31	-15.93
CAN	1970.1-2011.2	1.567	4.843	166.97	14.510	3.56	3.76	-18.55
FR	1973.2-2011.2	1.433	3.656	<0	18.340	4.58	5.02	17.80
GER	1978.4-2011.2	1.407	3.144	<0	8.010	4.95	5.19	-5.97
ITA	1971.2-2011.2	0.447	0.438	66.96	5.150	3.05	2.91	-8.73
JAP	1970.2-2011.2	0.864	1.867	118.09	6.320	2.96	3.62	-8.15
NTH	1977.2-2011.2	2.108	4.402	141.29	18.900	8.62	12.99	-8.86
SWT	1982.2-2011.2	2.308	5.296	483.74	35.600	3.15	2.79	-15.24
SWD	1920-1998	1.903	3.432	74.062	12.400	5.55	6.68	-13.17
UK	1919-1998	1.814	11.050	41.233	14.483	5.73	16.64	-11.75
USA	1891-1998	1.988	3.120	22.827	11.293	5.84	6.86	-11.25
	Median value	1.567	3.432	96.08	12.400	4.58	5.02	-11.25

$\gamma_m = \frac{E(r_m)-r_f}{\sigma_m^2} + \frac{1}{2}$, $\gamma_d = \frac{E(r_m)-r_f}{\sigma_d^2} + \frac{1}{2}$, and $\gamma_{CBAP} = \frac{E(r_m)-r_f+\frac{\sigma_m^2}{2}}{\sigma_c\sigma_m}$ is RRA 1 $\gamma_{2CBAP} = \frac{E(r_m)-r_f+\frac{\sigma_m^2}{2}}{\sigma_c\sigma_m}$ is RRA2 in Campbell (2003, Table 4) and Campbell (2017, Table 6.2). $TPR_c \equiv \delta_c = r_f - \gamma_c\mu_d + \frac{1}{2}\gamma_c^2\sigma_m^2$,$TPR_d \equiv \delta_d = r_f - \gamma_d\mu_d + \frac{1}{2}\gamma_d^2\sigma_m^2$, δ_{CBAP} is TPR1 TPR_{2CBAP} is TPR2 in Table 5 from Campbell (2003) and Campbell (2017), Table 6.3.

on the risk free rate, with the general result that an increase in γ lowers the risk free rate. It is useful to compare these results to other studies, either experimental or ones that use option prices, as summarized in the Table 3:

Although the methods used in the above papers are very different from the one used here, they produce results that at least overlap, in stark contradiction to the *CBAP* approach. The results in Table 2 also shed some light into the critique by Rabin (2000), that a small risk aversion for small bets would translate into absurdly large levels of risk aversion for large bets.

Table 3: γ and δ estimates in other studies

Study	γ	δ
Holt and Laury (2002)	$0.69 - 0.97$	
Bliss and Panigirtzoglou (2004)	$1.97 - 3.37$	
Andersen et al. (2008)	0.74	10.1

Sources: Holt and Laury (2002) Tables 3 and 4, for 90x high real choices, Bliss and Panigirtzoglou (2004) Table V, 6 week forecast horizon, FTSE and S&P 500 Options. Andersen et al. (2008), Table III

4 Pricing and returns of individual securities

With the estimates of the preference parameters, we could recalculate the Arrow-Debreu state prices. For example, using the U.S. data, we have:

$$p_s = \pi_s exp\left(-r_f - 1.976\sigma^2 - 1.988\sigma s\right)$$

One could use readily observable forward looking market information such as r_f and VIX. Any arbitrary asset with cash flows cf_{js} is worth $V_j = \int p_s cf_{js} ds$. This valuation method, while mathematically straightforward and completely general in nature, lacks a good deal of economic intuition. To study one specific case of interest, consider a corporation j that produces the following cash flows:

$$cf_{js} = a_j + b_j E_s^{\beta_j} + \varepsilon_{js}$$

Where ε_{js} is an idiosyncratic risk that is uncorrelated to the stochastic discount factor, i.e., has $E(\varepsilon_{js}) = E(\varepsilon_{js} E_s^{-\gamma}) = 0$. I will show that while it is very easy to price this asset, it is more complicated to state its expected returns. Let us begin by looking at the value of the complex security:

$$V_j = a_j e^{-\delta} E\left[\left(\frac{E_s}{E_0}\right)^{-\gamma}\right] + b_j E_0^{\beta_j} e^{-\delta} E\left[\left(\frac{E_s}{E_0}\right)^{\beta_j - \gamma}\right] + \frac{e^{-\delta}}{E_0^{-\gamma}} E\left[\varepsilon_{js} E_s^{-\gamma}\right]$$

$$V_j = a_j e^{-\delta - \gamma\mu + \frac{\gamma^2\sigma^2}{2}} + b_j E_0^{\beta_j} e^{-\delta} e^{(\beta_j - \gamma)\mu + \frac{1}{2}(\beta_j - \gamma)^2\sigma^2}$$

$$V_j = a_j e^{-r_f} + b_j E_0^{\beta_j} e^{-\delta} e^{(\beta_j-\gamma)\mu+\frac{1}{2}(\beta_j-\gamma)^2\sigma^2} \equiv V_{dj} + V_{ej}$$

This equation exploits the fact that the a_j (which can be negative) cash flows look like a riskless zero coupon bond, that $ln\left(\frac{E_s}{E_0}\right)^{\beta_j-\gamma} \sim N((\beta_j - \gamma)\mu, (\beta_j-\gamma)^2\sigma^2)$, and that $\int_{s\epsilon S} p_s \varepsilon_{js} = \frac{e^{-\delta}}{E_0^{-\gamma}} E(\varepsilon_{js} E_s^{-\gamma}) = 0$. This last result means that the economy does not subtract value from idiosyncratic risks, just as in the standard CAPM model by Sharpe (1964). The value V_j can be partitioned into a riskless debt component worth V_{dj} and an equity component worth V_{ej}. The expected simple rate of return for security j is $E(\widetilde{r}_j)$, and it looks like a weighted average cost of capital, i.e.:

$$E(\widetilde{r}_{js}) = \frac{V_{dj}}{V_j}\widetilde{r}_f + \frac{V_{ej}}{V_j}E[\widetilde{r}_{ejs}]$$

Where $\widetilde{r}_f = e^{r_f} - 1$. It is a well known result by Stiglitz (1969) that in a complete market with no bankruptcy costs, the Modigliani Miller proposition I holds, i.e. the enterprise value does not vary with changes in capital structure, even with risky debt. This implies that the Modigliani Miller proposition II also holds, i.e. that $E(\widetilde{r}_j)$ is invariant to changes in capital structure. Hence, it is valid to look at this specific capital structure to obtain the expected return on security j. I will now focus on the equity-like part of the cash flows, and look at the simple systematic returns:

$$\widetilde{r}_{ejs} = \frac{b_j E_s^{\beta_j}}{V_{ej}} - 1 = \frac{b_j E_0^{\beta_j}\left(\frac{E_s}{E_0}\right)^{\beta_j}}{b_j E_0^{\beta_j} e^{-\delta} e^{(\beta_j-\gamma)\mu+\frac{1}{2}(\beta_j-\gamma)^2\sigma^2}} - 1$$

For the following calculations it is easier to work with continuously compounded rates of return:

$$r_{ejs} \equiv ln(1+\widetilde{r}_{ejs}) = \delta - (\beta_j-\gamma)\mu - \frac{1}{2}(\beta_j-\gamma)^2\sigma^2 + \beta_j\mu + \beta_j\sigma s$$

This equation can be simplified using equations (2) and (3) to obtain the following result:

$$r_{ejs} = r_f + \beta_j(r_{ms} - r_f) + \frac{\sigma^2}{2}\beta_j(1 - \beta_j)$$

In expected returns, the above equation would yield:

$$E(r_{ejs}) = r_f + \frac{\sigma^2}{2}\beta_j(1 - \beta_j) + \beta_j \left[E(r_{ms}) - r_f\right] \tag{5}$$

If we define excess expected returns as $z_s \equiv r_s - r_f$, the above equation becomes:

$$E(z_{ejs}) = \frac{\sigma^2}{2}\beta_j(1 - \beta_j) + \beta_j E(z_{ms}) \equiv \alpha_j + \beta_j E(z_{ms})$$

Where $\alpha_j \equiv \frac{\sigma^2}{2}\beta_j(1 - \beta_j)$ and σ^2 is the market volatility. This equation implies that for $\beta_j < 1$ then $\alpha_j > 0$, and for $\beta_j > 1$, then $\alpha_j < 0$. This theoretical prediction is similar to the CAPM in Black (1972), Merton (1973), and found in the earliest empirical studies of the model, such as Black, Jensen, and Scholes (1972). However, in our case, this is due to the log-normality of returns. To obtain the expected simple rate of return, re-arrange equation (5) as follows:

$$E(r_{ejs}) + \frac{\beta_j^2\sigma^2}{2} = r_f + \beta_j \left[E(r_{ms}) + \frac{\sigma^2}{2} - r_f\right]$$

remember that for a continuously compounded rate $r_{xs} \sim N(\mu_x,\sigma_x^2)$, then its simple counterpart has $E(1 + \widetilde{r}_{xs}) = 1 + E[\widetilde{r}_{xs}] = E(e^{r_{xs}}) = e^{\mu_x + \frac{\sigma_x^2}{2}}$, and $ln\,(1 + E[\widetilde{r}_{xs}]) = \mu_x + \frac{\sigma_x^2}{2}$, hence

$$ln(1 + E[\widetilde{r}_{ejs}]) = ln(1 + \widetilde{r}_f) + \beta_j \left[ln\,(1 + E[\widetilde{r}_{ms}]) - ln(1 + \widetilde{r}_f)\right] \tag{6}$$

Where $\beta_j = \frac{cov(r_{ejs},r_{ms})}{var(r_{ms})}$ with the continuously compounded returns. This logarithmic CAPM can be viewed in several ways:

1. The exponential of equation (6) yields:

$$1+E[\widetilde{r}_{ejs}]=(1+\widetilde{r}_f)\left(\frac{1+E[\widetilde{r}_{ms}]}{1+\widetilde{r}_f}\right)^{\beta_j}$$

2. If we define a discount factor rate $\hat{r}_x = ln(1+E[\widetilde{r}_{xs}])$ as one that establishes the present value of an expected cash flow, i.e. $V_x = e^{-\hat{r}_x}E(cf_x)$, then equation 6) becomes:

   $$\hat{r}_{ej}=r_f+\beta_j\left(\hat{r}_m-r_f\right)$$

   Where $\hat{r}_m = E(r_{ms})+\frac{\sigma^2}{2}=r_f+\gamma\sigma^2$.

3. If we replace the right hand side of the above equation with its fundamental formulas, found in section 3, we obtain:

   $$\hat{r}_{ej}=\delta+\gamma\mu-\frac{1}{2}\gamma^2\sigma^2+\beta_j\gamma\sigma^2$$

4. If we linearize equation (6) with a Taylor expansion such that $ln(a+x)\approx ln(a)+\frac{x}{a}\mid_{a=1}= x$, then Sharpe's CAPM can be seen as an approximation of equation (6):

   $$E[\widetilde{r}_{ejs}]\approx\widetilde{r}_f+\beta_j\left(E[\widetilde{r}_{ms}]-r_f\right)$$

I would like to end this section with a word of caution. For an empirical examination of these results, we need to distinguish between fundamental assumptions (used in sections 2 and 3) and auxiliary premises (used in this section to specify the company cash flows cf_{js}). The failure of any of these types of assumptions would lead to an empirical falsification of equation (6). Such a rejection would not be so damaging if I have simply mis-specified the cash flows, but it would be more consequential if it is due to the failure of the pricing from equation (1).

The proper formulation of corporate cash flows is an important area for future research. Indeed, papers by Mclean and Pontiff (2016), Harvey et al. (2016), and Hou et al. (2017) have shown that most of the 'factors' discovered in empirical asset pricing are due to either data mining, or to market inefficiencies that are quickly corrected. There are, however, some factors that seem to endure. It would be interesting to then look at the behavior of firms affected by such relevant factors, to explore if their cash flows depart in a significant way from the specification set forth here. If such a departure in cash flow specification is true, then the framework developed in sections 2 and 3 could still account for their pricing and expected returns.

5 A simple multi-period extension

A multi-period extension of the model in sections 2 and 3 can be done in continuous or discrete time with periods up to T. Consider the following: each investor $i = 1, ..., I$ is given endowments e_{ist}. Each period $t > 0$, there is an aggregate endowment E_{st}. Define the set of states of nature in period t as S_t, and any particular element as $s_t \epsilon S_t$, with a probability π_{st} with the condition that either $\sum_{s_t \epsilon S_t} \pi_{st} = 1$ for discrete states and $\int\limits_{s_t \epsilon S_t} \pi_{st} ds_t = 1$ for continuous states for every t. The generalized problem that each investor faces is:

$$L_i = u_i(c_{i0}) + \sum_{t=1}^{T} \left[e^{-t\delta_{it}} \int\limits_{s_t \epsilon S_t} \pi_{st} u_i(c_{ist}) ds_t \right] + \lambda_i \left[(e_{i0} - c_{i0}) + \sum_{t=1}^{T} \int\limits_{s_t \epsilon S_t} p_{st} \left(e_{ist} - c_{ist} \right) ds_t \right]$$

$$L_i = u_i(c_{i0}) + \int_0^T \left[e^{-t\delta_{it}} \int\limits_{s_t \epsilon S_t} \pi_{st} u_i(c_{ist}) ds_t \right] dt + \lambda_i \left[(e_{i0} - c_{i0}) + \int_0^T \int\limits_{s_t \epsilon S_t} p_{st} \left(e_{ist} - c_{ist} \right) ds_t dt \right]$$

The problems are stated in discrete and continuous time setups, with continuous states of nature. For each period we have power utility functions as in the two period case. I have allowed for the time preference rate δ_{it} to vary with time. The pricing solution is straightforward by exploiting the aggregation that comes if $\delta_{it} = \delta_t$ and $\gamma_i = \gamma$, and the general equilibrium properties of the problem. State prices are:

$$p_{st} = e^{-t\delta_t}\pi_{st}\left(\frac{E_{st}}{E_0}\right)^{-\gamma}$$

Any complex security j with cash flows cf_{jst} is worth $V_j = \sum_{t=1}^{T}\left(\int_{s_t \epsilon S_s} p_{st} cf_{jst}\right)$ for a discrete period framework, and $V_j = \int_0^T \left(\int_{s_t \epsilon S_s} p_{st} cf_{jst}\right) dt$ for its continuous counterpart.

To establish the properties of $\left(\frac{E_{st}}{E_0}\right)$, consider discrete time case. Suppose that change in aggregate endowments, conditional on information at time zero, has:

$$ln\left(\frac{E_{st}}{E_{st-1}} \mid \mathcal{F}_0\right) = \mu_t + \sigma_t s_t$$

Where $s_t \sim N(0,1)$ has a unit normal distribution. This implies that:

$$ln\left(\frac{E_{st}}{E_0}\right) = \sum_{\tau=1}^{t} ln\left(\frac{E_{s\tau}}{E_{s\tau-1}}\right) = \sum_{\tau=1}^{t}(\mu_\tau + \sigma_\tau s_\tau)$$

It is clear that $\frac{E_{st}}{E_0}$ is log-normally distributed, given $ln\left(\frac{E_{st}}{E_0}\right)$, the sum of normal shocks, is normally distributed. To present the simplest result, consider the situation where $\mu_t = \mu$, $\sigma_t = \sigma$ and s_t is independently distributed. In that case $E\left[ln\left(\frac{E_{st}}{E_0}\right)\right] = t\mu$ and $var\left[ln\left(\frac{E_{st}}{E_0}\right)\right] = t\sigma^2$. In summary, we have that $ln\left(\frac{E_{st}}{E_0}\right) \sim N(t\mu, t\sigma^2)$. The continuously compounded risk free yield for a zero coupon bond with maturity t is r_{ft} . The price of such bond is $V_{r_{ft}} = e^{-tr_{ft}}$, and using the results in equation (2) yields:

$$r_{ft} = \delta_t + \gamma\mu - \frac{1}{2}\gamma^2\sigma^2 \tag{7}$$

The yield curve in this simple case would only depend on the shape of δ_t, but this could of course be enriched by modeling more complex behaviors for μ_t, σ_t and auto correlations for s_t. For the expected market discount rate (expressed as per one unit of time), we would also obtain:

$$E(r_{mst}) = \delta_t + \gamma\mu - \frac{\sigma^2}{2}(\gamma - 1)^2 = r_{ft} + \gamma\sigma^2 - \frac{\sigma^2}{2} \tag{8}$$

The market risk premium is constant in this very simple specification of the multi-period model.

6 Conclusion

I have studied an Arrow Debreu economy with heterogeneous agents, additive power utilities that allow for aggregation, and log-normal aggregate endowments. These assumptions have been used before, but the approach here is to calculate asset prices and returns directly, rather than first solving the optimal consumption and investment decisions. Such an approach yields the following simple yet powerful results:

The most important result is that the Arrow-Debreu securities have a price given by $p_s = e^{-\delta}\pi_s\left(\frac{E_s}{E_0}\right)^{-\gamma}$, where $\frac{E_s}{E_0}$ are the aggregate endowments for investors, which are then assumed to be log-normally distributed. Using this formula, I find familiar and surprising results. The first familiar result is the formula for the continuously compounded risk free rate, which is the same as in consumption based models, i.e. $r_f = \delta + \gamma\mu - \frac{1}{2}\gamma^2\sigma^2$.

The continuously compounded expected market return, on the other hand, is $E(r_{ms}) = r_f + \gamma\sigma^2 - \frac{\sigma^2}{2}$, and so the estimate for the coefficient of relative risk aversion becomes $\gamma = \frac{E(r_{ms})-r_f}{\sigma^2} + \frac{1}{2}$ .The difference between the model here and the consumption based asset pricing comes from general equilibrium consideration of asset pricing. This implies that the volatility of the market portfolio is identical to the volatility of the log change of the aggregate endowments of those agents who determine security prices. I argue that using aggregate consumption is misleading if there is a significant part of the wealth that is not traded.

Using data from Campbell (2003, 2017) I obtain a median relative risk aversion value of 1.567 and 3.432 depending on the proxy used for σ. Using the above calculations also yields a median time preference rate 4.58% and

5.02%. These results re-unite research using financial data with experimental studies on risk aversion and impatience, such as Holt and Laury (2002) and Andersen et al. (2008).

In the case where the cash flows of an individual security can be written as $cf_{js} = b_j E_s^{\beta_j} + \epsilon_{js}$ with $E(\epsilon_{js}) = E(\epsilon_{js} E_s^{-\gamma}) = 0$, the expected return of that security is $ln(1 + E[\widetilde{r}_{ejs}]) = ln(1 + \widetilde{r}_f) + \beta_j \left[ln(1 + E[\widetilde{r}_{ms}]) - ln(1 + \widetilde{r}_f) \right]$, i.e. a logarithmic CAPM.

The model is extended to a multi-period setting, yielding the same insights as with the two period setup. This extension allows for a simple yield curve that depends on the time preference rates, and on the behavior of short versus long term expected endowment growth and volatility.

In addition to the above results, this model is easy to extend according to the needs of the researcher, and so clarify even more the inner workings of financial economics.

7 References

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