A MEAN-VARIANCE SYNTHESIS OF CORPORATE FINANCIAL THEORY

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IN RECENT YEARS the elaboration of portfolio theory has shattered the conventional partitions within the field of finance. While it has always been desirable, it is now possible to treat security valuation, asset expansion decision rules, and capital structure policies as derivatives of market equilibrium models under uncertainty. Additionally, these models provide benchmarks for measuring the efficiency of markets and investment performance. Portfolio theory, providing as it does, theories of individual choice of securities and the determination of their market prices, therefore comprises the theoretical substructure of finance. The objective of this essay is to demonstrate that an integration of much of the subject matter of finance is possible at a relatively introductory level. No attempt is made to cover all the applications of portfolio theory; I have rather concentrated on the contributions of the popular mean-variance theory[1] to corporate finance, and consequently this essay is divided into three parts treating the three major problems of corporate finance: security valuation, asset expansion, and capital structure, in that order.

Much of the theory, informally treated in the text with formal arguments banished to footnotes, is contained in the existing literature, in particular Sharpe [15] on security valuation, Mossin [11] on asset expansion and Stiglitz [16] on capital structure.[2] However, several results will not be found in the published literature: (1) development of mean-variance capital budgeting criteria for mutually exclusive projects, capital rationing, and mutually interdependent projects; (2) proof that although non-synergistic merging typically reduces the probability of bankruptcy, shareholders will nonetheless be indifferent; (3) proof of the Modigliani-Miller Proposition I with risky corporate debt and corporate taxation; (4) proof of the Modigliani-Miller Proposition II revised for risky corporate debt; (5) analysis of the separate effects of operating risk and financial risk on equity risk premiums; (6) analysis of the components of operating risk; and (7), in the Appendix, a relatively elegant proof of the mean-variance security valuation theorem.

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1. The state-preference theory, developed for example by Myers [12], from which this theory can be derived as a special case, is omitted from this synthesis. However, while empirical tests of the more general theory are lacking, they are available for the mean-variance theory in increasing abundance in recent years. Jensen [4] provides an excellent summary of these results. He concludes that the model in its simplest form fails to explain adequately the structure of security returns; however, slightly generalized forms of the model which do not destroy its basic features appear more promising.

2. Portions of several other papers are summarized in the text, including those of Hamada [3], Lintner [5], Modigliani and Miller [9], and Mossin [10].

I. Security Valuation

Let us start from the familiar mean-variance security valuation theorem that under certain assumptions[3] it follows that for any security j

$$E(R_j) = R_F + \lambda \operatorname{Cov}(R_j, R_M) \quad (1)$$

3. The most important assumptions are (1) its single-period context, (2) no restrictions on short-selling and borrowing, and (3) a perfect and competitive securities market. However, Fama [1] has demonstrated that even though an individual has a concave multiperiod utility function, he will nonetheless behave in the first period as if he possesses some concave single-period utility function. This theorem is significant since if security returns are assumed normally distributed and intertemporally statistically independent, equation (1) applies even in a multiperiod setting where R_j represents a first period rate of return. Nonetheless, the model remains incapable of valuing irregular or non-perpetual income streams over time and hence has not rigorously been applied to the analysis of dividend policy and capital budgeting projects with multiperiod receipts. Only if firms can in some way estimate the probability distribution of the market value of a project at the end of the first period (without knowing future discount rates) and sale of the project at that time does not result in synergistic losses will the mean-variance model be appropriate. However, this model should not be criticized too heavily on this account since the present failure of theorists to produce any multiperiod (i.e., permitting portfolio revision over time) security valuation model under uncertainty consistent with maximizing expected utility (see Hakansson [2]) is very likely the most pressing theoretical problem in the field of finance.

The assumption of a *perfect* securities market precludes personal or corporate taxes, brokerage fees, underwriting costs, bankruptcy penalties, or other types of transactions costs as well as indivisibilities of securities. Relaxation of this assumption provides no analytical complications provided the imperfection is confined to a proportional reduction (possibly different for different securities) in the rate of return on a security; that is, stochastic constant returns must prevail. Otherwise, the necessary first order conditions in the Appendix must be drastically revised. However, if certain imperfections are admitted (as we will do in the case of proportional corporate income taxes) the capital structure and merger irrelevancy propositions do not strictly hold. Bankruptcy penalties, though not proportional corporate income taxes, create an incentive to merge since mergers almost invariably diminish the probability of bankruptcy. However, proportional personal income taxes do not affect any of the conclusions in this essay.

With a *competitive* securities market, the same security investment opportunities are available to all investors and no investor believes he can influence the rate of return on any security by his market transactions. No such assumption is made for firms in Sections I and III. However, in Section II, a firm's capital budgeting decisions are assumed to have negligible impact on the capitalized opportunities of other firms. The implications of relaxing the assumption of a competitive securities market have received little attention in the theoretical literature.

Rubinstein [13] demonstrates that the assumption of (4) the existence of a risk-free (i.e. zero variance) security is not substantive provided at least two risky securities exist in which case the symbol R_F in this paper may be replaced at every point by $E(R_p$ where p is a portfolio with $\operatorname{Cov}(R_p, R_M) = 0$. The strong short-selling assumption, by circumventing the issue of personal bankruptcy, makes this possible. Restrictions on short-selling leading to Kuhn-Tucker conditions have been examined by Lintner [5,6].

If the assumption of (5) homogeneous subjective probabilities is omitted, as Lintner [6] has shown, a concept similar to λ remains well-defined. However expected rates of return and covariances must be replaced by weighted averages. Furthermore, the convenient separation property of the model (i.e. all individuals regardless of differences in wealth levels or preferences, divide their wealth between the same two mutual funds, one of which is risk-free and the other the market portfolio of risky securities) no longer holds. As Stiglitz [16] proves, this failure of the separation property invalidates the Modigliani-Miller Proposition I in the presence of risky corporate borrowing. However, if corporate debt is risk-free, the proposition still holds. A similar qualification applies to the asset expansion propositions; see Lintner [8] and Myers [12].

The assumption that (6) all individuals evaluate portfolios by only two parameters, expectation and variance of future wealth, if omitted leads to a more complex security valuation equation which nonetheless preserves many of the characteristics of the simpler mean-variance case; see Rubinstein [13]. However, in this case the separation property is more difficult to obtain. Finally, if the assumption of (7) risk aversion is omitted, equation (1) remains necessary but no longer sufficient for market equilibrium.

where R_j (random variable) is the rate of return on security j,
R_F is the rate of return on a risk-free security,
R_M (random variable) is the rate of return on the market portfolio of risky securities, and
λ is a positive constant.

See the Appendix for a short proof of this theorem. This market equilibrium relationship between security risk and return may be interpreted in perhaps more familiar language by defining $R_j \equiv \tilde{P}_j/P_j$ where P_j is the present price of security j and P_j (random variable) is the change in price of security j.[4] With this definition it follows immediately that

$$P_j = \frac{E(\tilde{P}_j)}{R_F + \lambda \operatorname{Cov}(R_j, R_M)} = \frac{E(\tilde{P}_j) - \lambda \operatorname{Cov}(\tilde{P}_j, R_M)}{R_F},$$

the first equality representing a risk-adjusted discount rate formula, the second equality a certainty-equivalent formula.[5] Equation (1) is illustrated graphically in Figure 1. Since λ and R_F are market parameters, all securities have risk and return characteristics which fall along the "λ market line" in equilibrium. Define α_j as the proportion of the value of an arbitrary portfolio p assigned to security j. Observing that $R_p = \Sigma_j \alpha_j R_j$, it is easily demonstrated that all possible portfolios of securities fall along this same market line;[6] that is, for any portfolio p,

$$E(R_p) = R_F + \lambda \operatorname{Cov}(R_p, R_M).$$

Further since the market portfolio of risky securities is itself a portfolio, its risk and return characteristics fall along the market line; that is,

$$E(R_M) = R_F + \lambda \operatorname{Cov}(R_M, R_M) = R_F + \lambda \operatorname{Var} R_M$$

or alternatively,

$$\lambda = \frac{E(R_M) - R_F}{\operatorname{Var} R_M}.$$

Two popular alternative formulations of the equation (1) are

4. If R_j is defined as a *rate* of return, $\tilde{P}_j$ must be interpreted as a perpetual flow. Alternatively, R_j can be regarded as *one plus* the rate of return in which case $\tilde{P}_j$ must be interpreted as the future price of security j, a stock variable. With this latter definition of $\tilde{R}_j$ all equations in the text remain unchanged; however, all flow variables must be regarded as stock variables and, in particular, τ_j must be considered as a *wealth* tax rate on equity. To see this, observe that equation (1) holds if and only if

$$E(1 + R_j) = (1 + R_F) + \lambda \operatorname{Cov}(1 + R_j, 1 + R_M).$$

5. In the traditional risk-adjusted discount rate and certainty-equivalent dividend capitalization equations, the relationship between the risk premium or certainty-equivalent factor in each period and the risk characteristics of the dividend stream are unspecified. Unless some relationship is postulated, the equations remain merely *definitions* of the risk premiums or certainty-equivalent factors. In this context, the contribution of the mean-variance security valuation *theorem* is to provide the needed specification of the risk premium and certainty-equivalent factor.

6. To see this, merely multiply equation (1) by α_j and take the summation over all j.

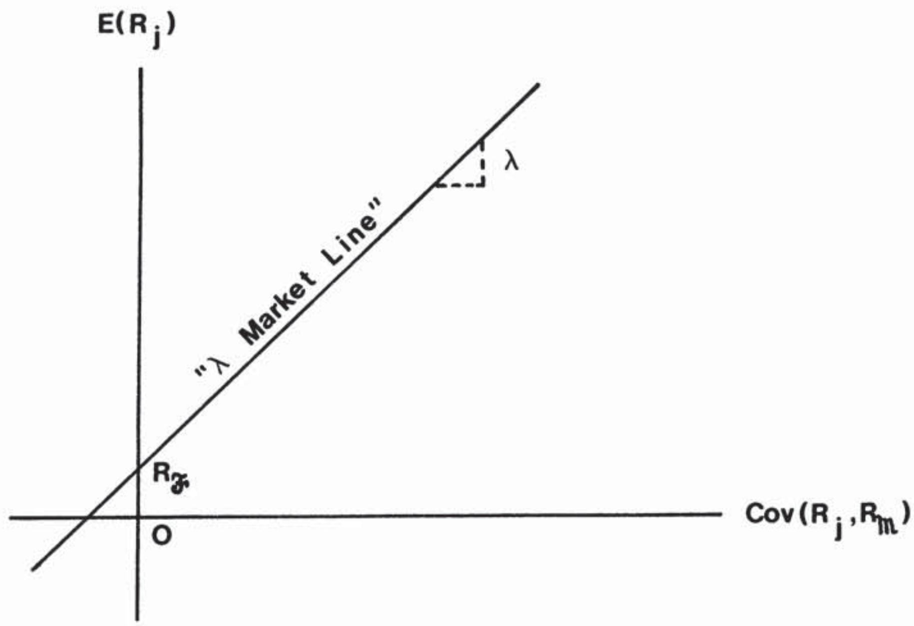

FIGURE 1

$$E(R_j) = R_F + \lambda^* \rho(R_j, R_M) \sqrt{\text{Var } R_j} \quad (2)$$

$$E(R_j) = R_F + \lambda^{**} \beta_j \quad (3)$$

where $\lambda^* \equiv \lambda \sqrt{\text{Var } R_M} = [E(R_M) - R_F] \sqrt{\text{Var } R_M}$,
$\lambda^{**} \equiv \lambda \text{ Var } R_M = E(R_M) - R_F$,
$\beta_j \equiv \text{Cov}(R_j, R_M)/\text{Var } R_M$, and
$\rho(R_j, R_M)$ is the correlation coefficient between R_j and R_M.

Unlike λ and λ^{**}, λ^* is dimensionless. These equations are derived from the definition of correlation coefficient and the result that the market portfolio of risky securities falls along the market line. These results can be described by similar graphical representations (see Figure 2). Equation (2) permits convenient distinctions between types of risk. Held alone as a portfolio, the risk of security j to an individual can be measured by $\sqrt{\text{Var } R_j}$; in a market or well-diversified portfolio context, the risk is measured by $\rho(R_j, R_M)\sqrt{\text{Var } R_j}$ (see equation (2)). The former may be called the *total risk* and the latter the *nondiversifiable* or *systematic risk*. Since $-1 \leqslant \rho(R_j, R_M) \leqslant 1$, $\rho(R_j, R_M)$ may be interpreted as the percentage of total risk that cannot be eliminated by diversification without sacrificing expected rate of return. The difference between total and nondiversifiable risk, $[1 - \rho(R_j, R_M)]\sqrt{\text{Var } R_j}$, measures the portion of the total risk that can be eliminated by diversification and hence can be called *diversifiable* or *nonsystematic risk*. All these results can, of course, be shown to hold for portfolios as well as securities.

If all diversifiable risk has been eliminated from a portfolio, we define that portfolio to be *efficient*. In this case

$$[1 - \rho(R_\epsilon, R_M)] \sqrt{\text{Var } R_\epsilon} = 0$$

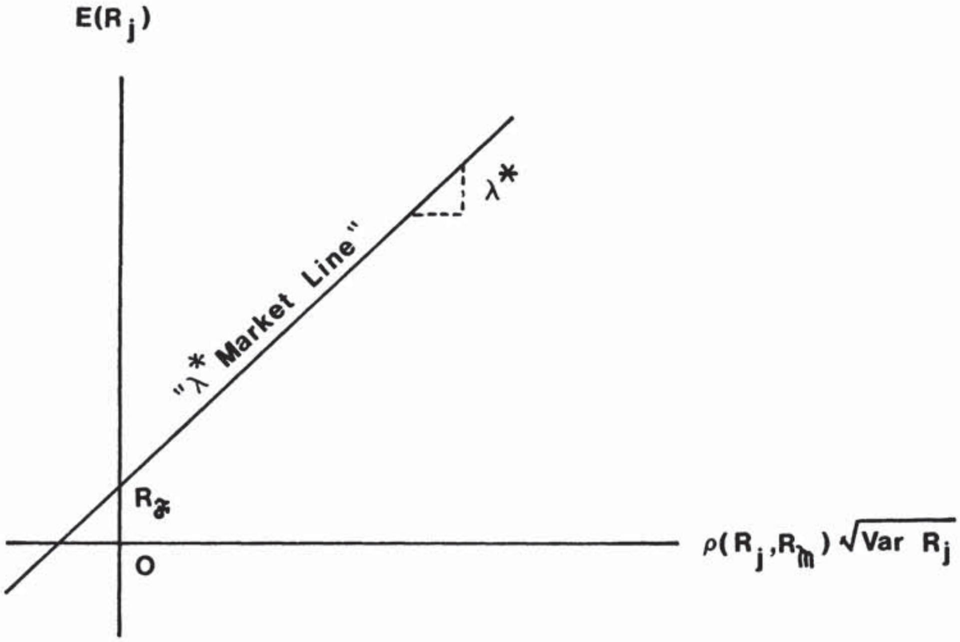

FIGURE 2

where subscript ϵ denotes an efficient portfolio. For this equation to hold, $\rho(R_\epsilon, R_M) = 1$; or, in other words, all efficient portfolios are perfectly positively correlated with the market portfolio of risky securities (with the exception of the efficient portfolio containing only risk-free securities). It follows that $\rho(R_p, R_M)$ may be interpreted as a dimensionless measure of the degree of diversification of any portfolio p. This analysis provides a method of segregating the efficient portfolios from other portfolios and securities which fall along the "λ^* market line" by noting that an "efficient portfolio market line" is described by setting the correlation coefficient equal to 1 in equation (2), and hence

$$E(R_\epsilon) = R_F + \lambda^* \sqrt{\text{Var } R_\epsilon}.$$

This, by the way, is the same line each individual will derive in a Markowitz efficient set analysis with the existence of a risk-free security and homogeneous subjective probabilities.

II. ASSET EXPANSION

The mean-variance security valuation theorem is readily applied to capital budgeting decisions for share price maximizing firms.[7] Consider j now as referring to firm j and R_j as representing the rate of return on the equity of firm j. It is easily demonstrated that firm j should accept a project only if

$$E(R_j^o) > R_F + \lambda \text{ Cov } (R_j^o, R_M)^8$$

7. The following theory does not actually require share price maximizing behavior for firms; it merely indicates the influence of capital budgeting decisions on share prices.

8. Alternative present value risk-adjusted discount rate and certainty-equivalent forms of this criterion are easily derived. Further, the criterion also has alternative formulations analogous to

where X_j^o (random variable) is the dollar return of the project,
$COST_j^o$ is the cost of the project,[9] and
$R_j^o \equiv X_j^o/COST_j^o$ (random variable) is the rate of return of the project.

This decision rule advises acceptance of a project only if its expected internal rate of return $E(R_j^o)$ exceeds the appropriate risk-adjusted discount rate for the project, $R_F + \lambda\,Cov(R_j^o, R_M)$; this discount rate is equal to the expected rate of return on a security with the same risk as the project. Graphically, in Figure 3, the acceptance criterion implies a firm should accept

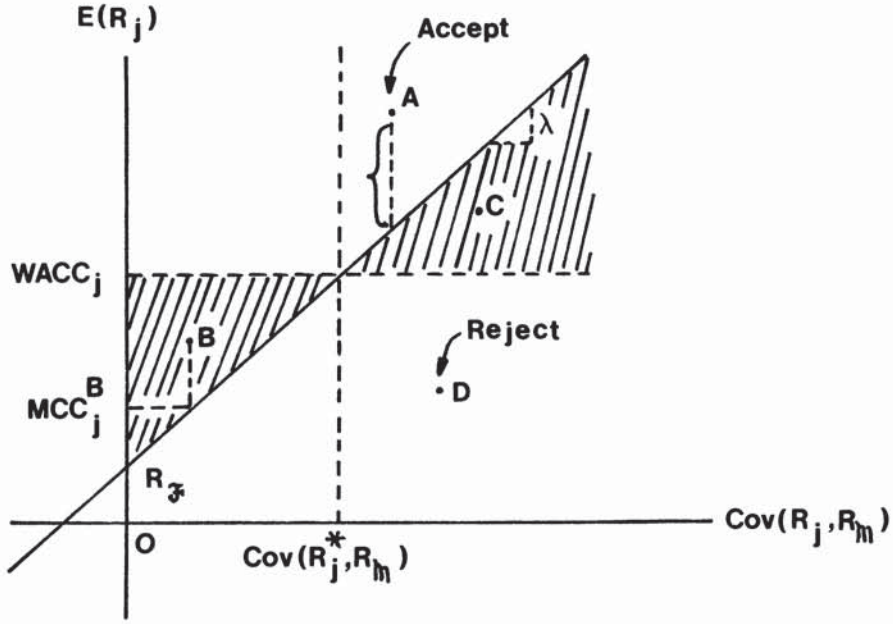

FIGURE 3

a project only if the *project's* risk-return order pair plots above the market line, such as projects A and B. In this case, when the firm accepts such favorable projects, there must be an upward revision of the firm's share price. To see this, after acceptance of a favorable project but before the price adjustment, the firm can be viewed in temporary disequilibrium with the firm's risk-return ordered pair temporarily plotting above the line. To restore equilibrium, individuals cause $E(R_j)$ to be lowered to the market line by bidding up the share price of firm j.[10]

equations (2) and (3). However, the version given by Mossin [11, p. 755] similar to $[E(R_j^o) - R_F]/Cov\ (R_j^o, R_M) > \lambda$ is only correct provided $Cov\ (R_j^o, R_M) > 0$.

9. X_j^o and $COST_j^o$ should be understood to represent, respectively, the entire marginal dollar return and cost of the project to the firm. Synergistic benefits could clearly cause the same project to have different marginal dollar returns and costs to different firms. Again, X_j^o may be interpreted as a perpetual flow of income or as the future market value of the project, with corresponding interpretations for R_j^o; see footnote 4.

10. The asset expansion criterion is demonstrated formally under the convenient, though unnecessary, assumption of all-equity financed firms and projects. Consider firm j for which

The constant slope of the market line, λ, may be interpreted as the *risk-standardized cost of capital* appropriate to all firms and all projects since, if $\text{Cov}(R_j^o, R_M) > 0$, then a firm should accept a project only if

$$\frac{E(R_j^o) - R_F}{\text{Cov}\,(R_j^o, R_M)} > \lambda.$$

We will call this result the "market price of risk" (MPR) asset expansion criterion. All firms in the economy may use λ as a cutoff value for all projects;[11] this contrasts with the traditional "weighted average cost of capital" (WACC) criterion which must be computed separately for each firm, and as we will shortly show, is generally invalid. Further, since the contribution of the project to the firm's variance of equity rate of return does not affect the accept or reject decision given by the MPR criterion, *diversification* (i.e., reduction of Var R_j) can be ignored in capital budgeting decisions. That is, in the absence of *synergy* (i.e., if $R_j^o = R^o$, the rate of return if the project were itself a firm), each project is evaluated on its own merits without reference to the firm's existing investments.[12] This conclusion also follows from the observation that individuals, by their own diversification, can costlessly eliminate any diversifiable risk present in a firm's investment portfolio so that the firm need not diversify for individuals.[13]

P'_j is the revised present price after acceptance of the new project,
N_j is the number of shares before acceptance of the new project,
N_j^o is the additional shares issued at price P'_j to finance the project, and
X_j (random variable) is the dollar value of net operating income before acceptance of the new project.

From equation (1), before acceptance of the project since $R_j = X_j/(N_jP_j)$

$$E(X_j) = R_F N_j P_j + \lambda\, \text{Cov}\,(X_j, R_M). \quad \text{(a)}$$

After acceptance of the project

$$E(X_j + X_j^o) = R_F(N_j + N_j^o)P'_j + \lambda\ \text{Cov}\,(X_j + X_j^o, R_M). \quad \text{(b)}$$

Since by definition $X_j^o = R_j^o(\text{COST}_j^o)$ and $\text{COST}_j^o = N_j^o P'_j$, from equations (a) and (b)

$$R_F N_j(P_j - P'_j) = (\text{COST}_j^o)\,[R_F + \lambda\ \text{Cov}\,(R_j^o, R_M) - E(R_j^o)]. \quad \text{(c)}$$

The asset expansion criterion follows since both $R_F N_j$ and COST_j^o are positive. This analysis, however, ignores second order effects on R_M, and hence on λ. In U.S. capital markets such effects are likely to be insignificant. See Myers [12, pp. 12,13] for a more probing discussion of this point.

11. λ, therefore, is an important economy-wide variable. Several studies have attempted to measure the related λ^* from *ex-post* data; and the comparative statics analyses of Lintner [7] and Rubinstein [14] permit a theoretical examination of the determinants of λ^* and its behavior over time.

12. Since R_M reflects the existing investments of firm j as well as all other firms, this statement is not formally accurate. However, in U.S. capital markets the influence of a single firm's investments on R_M is likely to be insignificant.

13. The "homemade diversification" theorem should be regarded as one of the major discoveries in corporate financial theory. Despite lack of recognition in several recent papers, the theorem was first formally proven by Mossin [10, pp. 779-781]. Myers [12] has demonstrated a similar proposition with a state-preference model under complete (i.e., Arrow-Debreu) markets for securities and under incomplete (i.e., generalized) markets with the existence of security risk classes.

A common interpretation of the WACC criterion can easily be shown to be generally invalid. This interpretation advises that a project should be accepted only if $E(R_j^o) > WACC_j$, that is, graphically, only if it falls above the horizontal dotted line in Figure 3, such as projects A and C. Therefore, for projects that fall in the shaded areas, such as B and C, the WACC and MPR criteria lead to contradictory decisions. The WACC criterion is obviously invalid because it fails to consider the risk of projects. For example, projects with $E(R_j^o) > WACC_j$ but with very high risk, such as C, will be improperly accepted. In fact, the WACC criterion will only lead to the correct cut-off rate for projects in the same "risk class" as the firm; that is, projects for which $Cov(R_j^o, R_M) = Cov(R^*_j, R_M)$ where R^*_j (random variable) is the rate of return shareholders would earn if the firm kept its existing investments intact but altered its capital structure so that it became debt free. $Cov(R^*_j, R_M)$ therefore reflects only the "business" or "operating risk" of the firm as distinct from its "financial risk." In Section III, we will show that the ordered pair $(Cov(R^*_j, R_M), WACC_j)$ falls on the market line. Graphically, a project will be in the same "risk class" as the firm only if it plots on the vertical dotted line in Figure 3, from which it can be visually verified that only for such projects will the WACC criterion provide the appropriate cut-off rate. A second explanation of the failure of the WACC criterion is that it is not a marginal criterion. The appropriate marginal cost of capital (MCC_j^B) for project B is indicated on the vertical axis in Figure 3. The MCC_j^o depends on the risk of a project and is equal to the appropriate discount rate for the project, $R_F + \lambda\, Cov(R_j^o, R_M)$.

Mutually exclusive projects, capital rationing and mutually interdependent projects are easily treated in this framework. Suppose projects A and B in Figure 3 are mutually exclusive. It can be readily shown that the firm should accept the project with the highest excess expected internal rate of return weighted by its cost,[14] that is, the highest $(COST_j^o)[E(R_j^o) - R_F - \lambda\, Cov(R_j^o, R_M)]$. Note that the excess expected internal rate of return for project A, for example, is measured graphically by the vertical distance between A and the market line in Figure 3. With capital rationing, the proper procedure is again to reject all projects falling below the line. Of the remaining projects consider all possible bundles of projects satisfying the rationing constraint. These bundles are in effect mutually exclusive; therefore the firm should accept the bundle with the highest excess expected internal rate of return weighted by its cost. If n superscripts projects in a feasible bundle, this is equivalent to accepting that bundle for which $\Sigma_n(COST_j^n)[E(R_j^n - R_F - \lambda\, Cov(R_j^n, R_M)]$ is the highest. With mutually interdependent projects, selection is determined by appropriately increasing the number of mutually exclusive projects. Assume, for example, that projects A and C are mutually interdependent; in this case in addition to projects A and C treated separately, the joint acceptance of A and C is considered as a single project.

Decision rules regarding mergers are easily derived. Consider firms j and

14. This result follows immediately from equation (c) of footnote 10 and is equivalent to accepting the project with the highest net present value.

k which are contemplating merger. Let X_j, X_k, and X_{jk} (random variables) be the net operating income (EBIT) of firms j, k, and the pro-forma post-merger firm, respectively. In the absence of *synergy,* that is, if $X_{jk} = X_j + X_k$, the post-merger firm can be considered equivalent to a portfolio containing all the securities of firms j and k. Since all possible portfolios, as well as securities, fall along the market line, the post-merger firm will fall along the market line and it will be impossible for the shareholders of both j and k to benefit from the merger. Again, the intuitive reason for this result is costless "homemade diversification." Alternatively, since an individual could have held the stock of both j and k in his portfolio before the merger, his portfolio becomes no more diversified if he holds the post-merger firm.[15] Therefore, as in capital budgeting, the diversification effects of a merger will not affect equity values. However, mergers with synergy will affect equity values and such mergers can be analyzed similarly to capital budgeting projects, since with synergy, *from the point of view of firm j,* the risk-return ordered pair of firm k may not plot on the market line.

It has been argued (see Lintner [8, p. 107]) that with risky corporate borrowing, merging decreases the probability of bankruptcy (provided the separate net operating incomes of the merging firms are not perfectly correlated). As a result, the merged firm can borrow on more favorable terms thereby increasing the value of equity. However, an offsetting consequence of merging is overlooked in this argument. The effects of bankruptcy are double-edged since mergers also have the unfavorable consequence of removing the separated limited liabilities of the merging firms. Consider two firms j and k: prior to merger bankruptcy of k does not affect the returns to shares of j; subsequent to merger, the returns from j's portion of the merged firm would be reduced by the requirement of meeting k's defaulted portion of the merged firm's obligations.[16]

15. Before the merger, the shares of firms j and k could have been held in any proportion in an individual's portfolio; however, after the merger, shares in firms j and k can, in effect, be held only in a fixed proportion. It may be argued, therefore, that mergers destroy opportunities for individual portfolio selection and hence even a merger with positive synergy could reduce equity values. However, the portfolio separation property of the mean-variance model insures that relevant opportunities will remain intact.

16. The irrelevancy of mergers is formally demonstrated even where corporate debt is explicitly risky. Define R_{F_j}, R_{F_k}, and $R_{F_{jk}}$ (random variables) as the borrowing rates for firms j, k, and jk. To reflect the influence of bankruptcy, for example, the probability distribution of R_{F_j} could be defined such that $\bar{R}_{F_j} \equiv R_{F_j}$ if $X_j \geqslant \bar{R}_{F_j} B_j$ and $R_{F_j} \equiv X_j/B_j$ if $X_j \leqslant \bar{R}_{F_j} B_j$ where $\bar{R}_{F_j}$ is the promised or contracted rate of interest (see text for definitions of B_j and V_j). From equation (1) and since $R_j = (X_j - R_{F_j} B_j)/(V_j - B_j)$, $E(X_j) - B_j E(R_{F_j}) = R_F V_j - B_j R_F + \lambda \text{ Cov}(X_j, R_M) - B_j \lambda \text{ Cov } (R_{F_j}, R_M)$. Since risky debt is also a security, $E(R_{F_j}) = R_F + \lambda \text{ Cov } (R_{F_j}, R_M)$; therefore

$$E(X_j) = R_F V_j + \lambda \text{ Cov } (X_j, R_M). \quad \text{(a)}$$

Similar arguments may also be made for firms k and jk, so that

$$E(X_k) = R_F V_k + \lambda \text{ Cov } (X_k, R_M) \text{ and (c) } E(X_{jk}) = R_F V_{jk} + \lambda \text{ Cov } (X_{jk}, R_M). \quad \text{(b)}$$

These equations will hold regardless of the effects of reduced probability of bankruptcy on $R_{F_{jk}}$. Since $X_{jk} = X_j + X_k$, adding equations (a) and (b) and comparing the sum to equation (c) yields the result $V_{jk} = V_j + V_k$.

III. Capital Structure

The mean-variance security valuation theorem is also readily applied to the effect of capital structure on the value of a firm. Consider firm j for which

X_j (random variable) is the dollar value of net operating income,
B_j is the present dollar value of debt,
S_j is the present dollar value of equity,
$V_j \equiv B_j + S_j$ is the present total dollar value of the securities of the firm, and
$R_j \equiv (X_j - R_F B_j)/S_j$ (random variable) is the rate of return on equity.

Define these variables for a second firm denoted by a * superscript for which $B^*_j = 0$. Since the risk-return ordered pairs of all securities fall along the market line,

$$\frac{E(R_j) - R_F}{\rho(R_j, R_M)\sqrt{\text{Var } R_j}} = \frac{E(R^*_j) - R_F}{\rho(R^*_j, R_M)\sqrt{\text{Var } R^*_j}} = \lambda^*. \tag{4}$$

If $X_j = X^*_j$, it follows that $V_j = V^*_j$ by substituting the definitions of R_j and R^*_j in equation (4).[17] Interpreting this result, as a firm alters its capital structure, but before price adjustment, the firm moves along, not off, the market line achieving the precise risk-return trade-off which leaves the market indifferent and hence its stock price unchanged.[18]

17. This proposition is demonstrated formally in the presence of corporate taxes and risky corporate debt where superscript denotes after tax variables, τ_j is the corporate income tax rate for firm j, and R_{F_j} (random variable) denotes the rate of return on the risky debt of firm j. Since $E(R_j) = R_F + \lambda \text{ Cov } (R_j, R_M)$ and $R_j \equiv (X_j - R_{F_j}B_j)(1 - \tau_j)/S_j$, it follows in the levered case that

$$E(X_j)(1 - \tau_j) - E(R_{F_j})B_j(1 - \tau_j) = R_F S_j + \lambda(1 - \tau_j) \text{ Cov } (X_j, \hat{R}_M) - \lambda(1 - \tau_j)B_j \text{ Cov } (R_{F_j}, \hat{R}_M). \tag{a}$$

By similar reasoning since in the unlevered case $\hat{R}^*_j \equiv X_j(1 - \tau_j)/V^*_j$

$$E(X_j)(1 - \tau_j) = R_F V^*_j + \lambda(1 - \tau_j) \text{ Cov } (X_j, \hat{R}_M). \tag{b}$$

Further since risky debt is also a security, $E(R_{F_j}) = R_F + \lambda \text{ Cov } (R_{F_j}, R_M)$. Substituting this equation and equation (b) into equation (a) and recalling that by definition $V_j \equiv S_j + B_j$, it follows that $V_j = V^*_j + \tau_j B_j$. This is the familiar result of Modigliani-Miller which holds even the presence of risky debt.

It should be emphasized, as Stiglitz [16] has shown, for more general models which lack the separation property, the Modigliani-Miller Proposition I, with or without taxes, will not hold in the presence of risky debt. The separation property in the mean-variance model insures that changes in capital structure will not alter relevant opportunities available to individuals.

18. It is not difficult to demonstrate that if $V_j = V^*_j$, then $P_j = P^*_j$ where P_j and P^*_j refer to share prices. Imagine that a firm is first in an unlevered position so that $V^*_j = S^*_j = N^*_j P^*_j$ where N^*_j is the number of shares. The firm now levers its capital structure without affecting its net operating income by purchase of ΔN shares of equity at price P_j (*a priori,* possibly different from P^*_j) and financing $(\Delta N)P_j$ with debt so that $(\Delta N)P_j = B_j$. Hence

$$V_j \equiv S_j + B_j = (N^*_j - \Delta N)P_j + (\Delta N)P_j = N^*_j P_j,$$

and since $V_j = V^*_j$, then $P_j = P^*_j$.

To show that the ordered pair $(\rho(R^*_j, R_M)\sqrt{\text{Var } R^*_j}, \text{WACC}_j)$ falls on the "λ^* market line" (or, alternatively, that $(\text{Cov}(R^*_j, R_M), \text{WACC}_j)$ falls on the "λ market line") we need only recall that by definition

$$\text{WACC}_j \equiv R_F\left[\frac{B_j}{V_j}\right] + E(R_j)\left[\frac{S_j}{V_j}\right] = \frac{E(X_j)}{V_j}$$

and since $X_j = X^*_j$ and $V_j = V^*_j$,

$$\text{WACC}_j = \frac{E(X_j)}{V_j} = \frac{E(X^*_j)}{V^*_j} = E(R^*_j).$$

Since R^*_j is independent of capital structure, it follows that the weighted average cost of capital is also.

The precise relationship between expected rate of return to equity and capital structure is easily demonstrated since by substitution of $R_j \equiv (X_j - R_F B_j)/S_j$ and $R^*_j = X_j/V_j$:

$$\rho(R_j, R_M) = \rho(R^*_j, R_M) \quad \text{and} \quad \sqrt{\text{Var } R_j} = \sqrt{\text{Var } R^*_j}\left[1 + \frac{B_j}{S_j}\right]. \tag{5}$$

Therefore, from equation (2),

$$E(R_j) = R_F + \lambda^*\rho(R^*_j, R_M)\sqrt{\text{Var } R^*_j}\left[1 + \frac{B_j}{S_j}\right]. \tag{6}$$

Equation (6) quantifies the effect of financial leverage on the risk of a firm and hence on its expected rate of return to equity. Equation (5) indicates that since the correlation coefficient is invariant to changes in financial leverage, the full impact of financial risk is absorbed by the standard deviation $\sqrt{\text{Var } R_j}$. Further, since both λ^* and $\sqrt{\text{Var } R^*_j}$ are positive, the direction of the influence of changes in financial leverage on $E(R_j)$ depends on the sign of $\rho(R^*_j, R_M)$, so that $E(R_j)$ could conceivably decrease with increased financial leverage. In more familiar terms, equation (6) is a specialization of the Modigliani-Miller Proposition II,

$$E(R_j) = E(R^*_j) + [E(R^*_j) - R_F]\left[\frac{B_j}{S_j}\right], \tag{7}$$

for the attitudes toward risk implied by the mean-variance security valuation theorem.[19] Both equations (6) and (7) permit separate analysis of operating risk and financial risk. Equation (6) can be written

19. If corporate debt is risky, equation (7) is generalized by replacing the symbol R_F with $E(R_{F_j})$. To see this, applying the definitions $R_j \equiv (X_j - R_{F_j}B_j)/S_j$, $R^*_j \equiv X_j/V^*_j$ and $V_j \equiv B_j + S_j$, it is easy to demonstrate that the revised equation (7) holds if and only if $V_j = V^*_j$, but this last equality has already been demonstrated in the absence of taxes in footnote 17. Since $E(R_{F_j})$ exceeds R_F for most firms in U.S. capital markets, the consideration of risky debt will cause $E(R_j)$ to be less than it would otherwise be if debt were assumed risk-free. This is intuitively plausible since with risky debt, the total risk of net operating income is shared by both equity and debt holders.

$$E(R_j) = R_F + \lambda^* \rho(R^*_j, R_M) \sqrt{\text{Var } R^*_j} + \lambda^* \rho(R^*_j, R_M) \sqrt{\text{Var } R^*_j} \left[\frac{B_j}{S_j} \right], \quad (8)$$

thereby separating the effects of the risk-free rate of return, operating risk, and financial risk on the expected rate of return to equity.[20]

It is further interesting to develop the components of operating risk. Consider, for firm j, product m for which

Q_m (random variable) is the output in units,
v_m is the variable cost per unit,
p_m is the sales price per unit,
F_m is the fixed cost, and
α_m is the proportion of assets (i.e., V_j) devoted to its production.

Therefore, assuming all fixed costs of the firm can be allocated, $X_j = \Sigma_m (Q_m p_m - Q_m v_m - F_m)$. Since $R^*_j = \alpha_m X_j / \alpha_m V_j$, it is not difficult to demonstrate that operating risk

$$\rho(R^*_j, R_M) \sqrt{\text{Var } R^*_j} = \Sigma_m [\alpha_m (p_m - v_m) \rho (Q_m, R_M) \sqrt{\text{Var } (Q_m/\alpha_m V_j)}]$$

where α_m measures the relative influence of each product line (assuming all assets of the firm can be allocated to products), $p_m - v_m$ reflects operating leverage, $\rho(Q_m, R_M)$ the pure influence of economy-wide events on output, and $\sqrt{\text{Var } (Q_m/\alpha_m V_j)}$ the uncertainty of output per dollar of assets which could be interpreted as a measure of the uncertainty of "operating efficiency."

Illustration of the effect of corporate income taxes on the relationship between capital structure and firm values is easily analyzed by equation (4) upon substitution of

20. This result can be used to explain the size of observed *ex post* values of β_j in equation (3). Defining $\beta^*_j \equiv \text{Cov } (R^*_j, R_M)/\text{Var } R_M$, it can be shown with adjustment for corporate income taxes that

$$\beta_j = \beta_j^* \left[1 + \frac{B_j(1 - \tau_j)}{S_j} \right].$$

From data in *Moody's Handbook of Common Stocks:* First Quarter, 1971, on General Motors (GM) and Chrysler (C) for 1960-1969, $\beta^*_{GM} = .77$ and $\beta^*_C = 1.69$ with $\beta_{GM} = .86$ and $\beta_C = 2.48$. We might infer that not only was Chrysler's "operating risk" about double General Motors' but the substantially higher financial leverage ratio for Chrysler ($B_{GM}/S_{GM} = .2$ and $B_C/S_C = 1.0$) caused Chrysler's nondiversifiable risk (operating plus financial) to be about triple General Motors'.

An alternative approach is to use equation (8) directly. The results for General Motors and Chrysler (adjusted for corporate income taxes) are summarized in the following table:

	$E(R_j)$	Risk-Free Rate R_F	Operating Risk $\lambda^* \rho(R^*_j, R_m) \sqrt{\text{Var } R^*_j}$	Financial Risk $\lambda^* \rho(R^*_j, R_m) \sqrt{\text{Var } R^*_j} \left[\frac{B_j(1-\tau_j)}{S_j} \right]$
GM	.10	.045	.050	.005
C	.21	.045	.110	.055

$$R_j = \frac{(X_j - R_F B_j)(1-\tau_j)}{V_j - B_j} \text{ and } R^*_j = \frac{X_j(1-\tau_j)}{V^*_j}$$

where τ_j is the corporate income tax rate of firm j.[21] We immediately derive the familiar result $V_j = V^*_j + \tau_j B_j$. This result can be given a similar interpretation to the acceptance of a project with a risk-return ordered pair falling above the market line. After an increase in financial leverage, but before price adjustment, the risk-return ordered pair of the firm's equity moves temporarily above the market line. To restore equilibrium, individuals cause $E(R_j)$ to be lowered to the market line by bidding up the share price of the firm.

In contrast to Modigliani and Miller [9], whose ingenious "risk class" assumption insulated their partial equilibrium approach from a need to provide a theory of the market risk premium, at some sacrifice of generality (see footnote 3), the mean-variance market equilibrium model provides this theory. By straightforward extensions, the most important concepts of corporate finance can be demonstrated by use of virtually a single diagram. Furthermore, quantification of risk premiums supplies the key to practical implementation.

APPENDIX[22]

A short proof of the mean-variance security valuation theorem follows under the special case of quadratic utility.[23] In addition to the variables already used in this paper consider individual i for which

S_{ij} is his present dollar value holdings of risky security j,

B_i is his present dollar value holdings of risk-free securities,

$W_i = \Sigma_j S_{ij} + B_i$ is his present wealth,

$\tilde{W}_i = \Sigma_j S_{ij} R_j + B_i R_F$ (random variable) is his future wealth (interpreting R as 1 + rate of return), and

$U_i(\tilde{W}_i)$ is his twice continuously differentiable measurable utility of future wealth function where $U'_i > 0$.

21. The corporate income tax is assumed to be proportional and with full loss offset. See footnote 17 for formal proof for the more general case of risky corporate debt.

In general, with the introduction of taxes, while all securities still fall along the market line, the slope of the line will change; see Rubinstein [14].

22. This proof as well as other arguments in this essay utilize freely and without comment basic properties of expectation operators. Specifically, if X and Y are any two random variables, a and b are nonrandom parameters, and i is an index, then (1) $E(a + bX) = a + bE(X)$, (2) $E(\Sigma_i X_i) = \Sigma_i E(X_i)$, (3) $Var(a + bX) = b^2 \text{ Var } X$, (4) $Cov(X,Y) = Cov(Y,X)$, (5) $Cov(a + bX,Y) = b\, Cov(X,Y)$, (6) $Cov(\Sigma_i X_i,Y) = \Sigma_i Cov(X_i,Y)$, (7) $Cov(X,Y) = E(XY) - E(X)E(Y)$, and (8) if $b > 0$, $\rho(a + bX,Y) = \rho(X,Y)$. It is, of course, assumed that all random variables have finite variances (and hence finite means).

23. Mossin [10] provides a more general proof assuming only ordinal utility functions with future value portfolio mean and variance as arguments. In a more recent paper, Mossin [11] offers another proof which sacrifices generality by assuming all individuals have measurable quadratic utility functions for future wealth; however, Mossin's new proof has the virtue of simplicity and provides detailed information about the determinants of λ. Nonetheless, as this appendix demonstrates, his new proof is needlessly lengthy. For the mean-variance security valuation equation to be consistent with measurable utility, one can alternatively assume that all securities have normally distributed rates of return; see Rubinstein [14].

In this context, closure requires $S_j = \Sigma_i S_{ij}$ and $R_M = \Sigma_j S_j R_j / \Sigma_j S_j$. These definitions imply the simple Lagrangian form of optimization

$$\max_{\{S_{ij}\}, B_i} \quad E[U_i(\widetilde{W}_i)] + L_i[W_i - \Sigma_j S_{ij} - B_i]$$

with first order conditions[24]

$$E[U'_i R_j] = E[U'_i R_F] = L_i \quad \text{for all} \quad j \quad \text{and} \quad W_i = \Sigma_j S_{ij} + B_i.$$

These conditions imply $E[U'_i(R_j - R_F)] = 0$ which in turn implies $E(U'_i)E(R_j - R_F) + \text{Cov}\ (U'_i, R_j - R_F) = 0$. If $U_i = W_i - a_i W_i^2$ where a_i is a nonrandom parameter, then $U'_i = 1 - 2a_i W_i$ and therefore

$$[E(R_j) - R_F] \frac{E(U'_i)}{2a_i} = \text{Cov}\ (R_j, W_i).$$

Since this equation will hold for all individuals in the market,

$$[E(R_j) - R_F]\ \Sigma_i \frac{E(U'_i)}{2a_i} = \Sigma_i \text{Cov}\ (R_j, \widetilde{W}_i) = \text{Cov}\ (R_j, \Sigma_i \widetilde{W}_i).$$

Closure requires that $\Sigma_i \widetilde{W}_i = \Sigma_j S_j R_j + \Sigma_i B_i R_F = R_M \Sigma_j S_j + R_F \Sigma_i B_i$; however, since $\Sigma_j S_j$ and $R_F \Sigma_i B_i$ are nonrandom, $\text{Cov}\ (R_j, \Sigma_i W_i) = \text{Cov}\ (R_j, R_M \Sigma_j S_j) = (\Sigma_j S_j)\ \text{Cov}\ (R_j, R_M)$. Therefore,

$$E(R_j) = R_F + \left[\frac{\Sigma_j S_j}{\Sigma_i \frac{E(U'_i)}{2a_i}} \right] \text{Cov}\ (R_j, R_M);$$

and if the quantity in brackets is identified with λ,[25] then

$$E(R_j) = R_F + \lambda\ \text{Cov}\ (R_j, R_M).$$

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24. These first order conditions will be necessary and sufficient for a unique global maximum if $U''_i(\widetilde{W}_i) < 0$ and no security rate of return is perfectly correlated with the rate of return of any portfolio excluding it.

25. If $U''_i < 0$, then $2a_i > 0$. From this it follows that $\lambda > 0$ since both $\Sigma_j S_j$ and U'_i are assumed positive.

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