ARITHMETIC AND GEOMETRIC AVERAGES

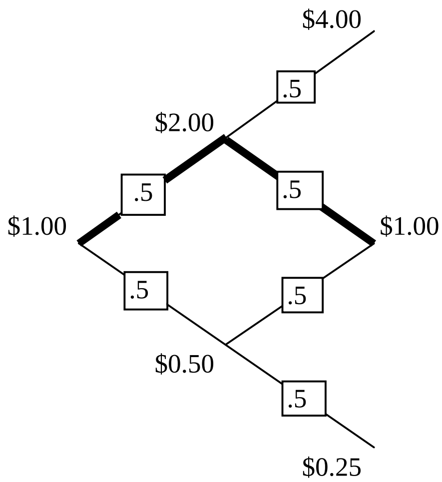
An arithmetic average of historical return data is the sum of all the periodic returns (the “period” is usually assumed to be one year), divided by the number of historical periods. A geometric average is a compound return—it is the rate of constant growth that would cause the security price at the beginning of the period to grow to the value realized at the end of the period.

The support for the use of an arithmetic mean of historical data rests in “decision tree” logic, which is demonstrated by the following example. Assume that an investor buys a stock for $1, and that stock has a 50% chance of doubling in price (increasing 100%) and a 50% chance of dropping by half (a loss of 50% of its value). Also assume that in the first year the stock price doubles from $1 to $2, but in the second year the stock price declines by 50%, resulting in a $1 price. The arithmetic average return is 25% [(100% + (-50%))/2 = 25%]. Because the investor winds up with $1 at the end of the second year after beginning with $1 at the outset, the geometric return is 0% [(1+100%)(1-50%) – 1 = 0%].

While it is counter-intuitive to state that the historical return in our example is 25% (the arithmetic average) when the investor winds up with the same amount of money at the end of two years as he or she began with, the rationale for the use of the arithmetic mean lies in the probabilities that existed for the investor at the outset. Those probabilities are best represented by the “decision tree” shown below, which displays all the possible outcomes for the investor (with the actual outcome designated by a bold line).

Chart I.

Decision Tree Example



In this example, the investors' expected return, which is calculated as the sum of all the possible outcomes, is $1.5625 [Expected Return = $(0.5)^2$($4.00) + $2(0.5)^2$($1.00) + ($0.5)^2$($0.25) = $1.5625]. The only way to calculate the $1.5625 value using historical average data is through the use of the arithmetic mean return [$1.5625 = $1.00(1.25)(1.25)]. This example provides support for the use of arithmetic averages of historical returns in estimating the cost of capital.

However, underlying the example cited above are some very strict assumptions about the relationship between year-to-year returns that are not representative of the actual nature of those returns. The "decision tree" assumes that the periodic returns are strictly independent results—each having no affect on the other. However, research indicates that such is not the case, and that period-to-period returns are inter-dependent to some degree.[1]

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[1] E. Fama and K. French, "Dividend Yields and Expected Stock Returns," *Journal of Financial Economics* (October 1988), pp. 3-26.

Therefore, the very strict “decision tree” logic used to support sole reliance on an arithmetic market risk premium does not apply to actual historical returns because those returns are inter-related and not strictly independent. Even academics that use arithmetic means of historical data recognize that if historical returns are not strictly independent (i.e., they are “serially correlated,” or are “mean reverting”), then the arithmetic mean does not provide a valid representation of the historical average return:

> If, however, the objective is to obtain the *median* future value of the investment, then the initial investment should be compounded at the geometric sample average. When returns are serially correlated, then the arithmetic average [footnote] can lead to misleading estimates and thus the geometric average may be the more appropriate statistic to use.
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> [footnote] The point is well illustrated by the textbook example where an initial investment of $100 is worth $200 after one year and $100 after two years. The arithmetic average return is 25% whereas the geometric average return is 0%. The latter coincides with the true return.[2]

Also, in a white paper presented to the Social Security Administration in 2001 regarding expected equity returns in the 21st Century, Professor John Campbell of Harvard provided the following comments regarding geometric means:

> When returns are negatively serially correlated, however, the arithmetic average is not necessarily superior as a forecast of long-term future returns. To understand this, consider an extreme example in which prices alternate deterministically between 100 and 150. The return is 50% when prices rise, and –33% when prices fall. Over any even number of periods, the geometric average return is zero, but the arithmetic average return is 8.5%. In this case the arithmetic average return is misleading because it fails

[2] (Mehra, R., Prescott, E., “The Equity Premium in Retrospect,” Handbook of the Economics of Finance, Constantinides, Harris, Stultz, Editors, 2003).

> to take account of the fact that high returns always multiply a low initial price of 100, while low returns always multiply a high initial price of 150. The geometric average is a better indication of long-term future prospects in this example. [footnote omitted]
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> The point here is not just a theoretical curiosity, because in the historical data summarized by Siegel, there is strong evidence that the stock market is mean-reverting. That is, periods of high returns tend to be followed by periods of lower returns. This suggests that the arithmetic average return probably overstates expected future returns over long periods.[3]

Finally, there are data anomalies associated with arithmetic risk premiums. The arithmetic market risk premium is period-specific. That is, the longer the assumed holding period, the lower the arithmetic risk premium. It is commonly assumed that the holding periods (the amount of time between buying and selling the market portfolio) is one year. However, there is no magic to that particular time-span, it is simply a common assumption in the calculation. If, for example, we assume that the holding period is two years instead of three, the arithmetic average market risk premium reported by Morningstar declines by 100 basis points. If that holding period increases to three years, the market risk premium declines again.[4] Therefore, the arithmetic mean changes with a change in the length of the holding period. The geometric mean does not vary with the holding period chosen, since the beginning and ending points determine the rate of growth.

In sum, both arithmetic and geometric averages have academic support in analyzing historical return data, and both should be considered in determining the cost of equity capital.

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3 (Estimating the Real Rate of Return on Stocks Over the Long Term, Papers by Campbell, Diamond, Shoven, Presented to the Social Security Advisory Board, August 2001; Cambell, J., "Forecasting U.S. Equity Returns in the 21st Century", pp. 3, 4).

4 Copeland, Koller, and Murrin, Valuation: Measuring and Managing the Value of Companies, 3rd Ed., McKinsey & Co., New York, 2006, pp. 218-221.